10.2 The Occurrence of Critical Flow; Controls

In addition to the type of problem in which both q and E are initially prescribed; there is a problem which is of practical interest: Given a value of q, what factors determine the specific energy E, and hence the depth y? Conversely, if E is given, what factors determine q?

The answer to these questions is that there are many different kinds of control mechanism which can dictate "what depth must be for a given q, and vice versa". Example is the sluice gate; For a given opening of the gate there is a certain relationship between q and the upstream depth, similarly for the downstream depth. Weirs and spillways are further examples of this kind of mechanism. The flow resistance due to the roughness of the channel bed will have some effect.

The flow situation in any channel is substantially influenced by the control mechanisms operating within it. The notion of a "control" - any feature which determines a depth - discharge relationship - is of primary importance in the study of free surface flow. There are certain features in channel which tend to produce critical flow, and are therefore controls (see box) of a rather special kind.

Three types of controls namely

(i) downstream control

(ii) upstream control and

(iii) Artificial control.

are identified.

Normally, the sub critical flow deals with downstream control and supercritical flow deals with the upstream control.

The nature of these features, are determined by considering the general problem of flow without losses in a rectangular channel section of constant width, whose bed level may vary. This is a particular situation of the transition problem. (See box).

Transition (flow basis):

1. Sub critical to Sub critical

2. Sub critical to Super critical (Hydraulic drop)

3. Super critical to Sub critical (Hydraulic Jump)

4. Super critical to Super critical

Transition Structure:

Converging Diverging

1. Rectangular cross section to Rectangular cross section

2. Rectangular cross section to Trapezoidal cross section

3. Trapezoidal cross section to Trapezoidal cross section

4. Trapezoidal cross section to Rectangular cross section

5. Trapezoidal cross section to circular cross section or Horse shoe tunnel

6. Horse shoe tunnel to Trapezoidal cross section

7. Horse shoe tunnel to Rectangular cross section etc.

Method of connection in transition (gradual):

a. By straight wall

b. By Quadrant (cylindrical)

c. By warped

The transition could be abrupt such as sudden expansion or sudden contraction.

The transition could be gradual over certain distance.

The transition can be in vertical plane such as steps, humps, drops.

The transition could be both in plan and in elevation.

Generally, transition are provided as inlet and outlet structure.

The flow in such transitions is three dimensional and complex.

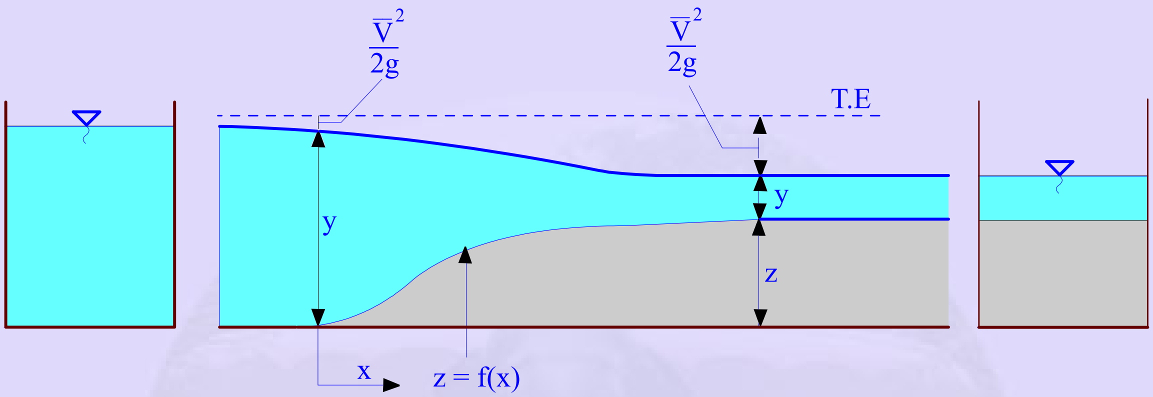
Following assumptions are made

1. Constant rectangular channel

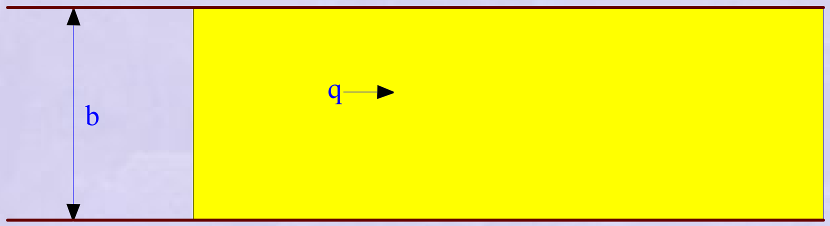
2. Short reach.

3. No frictional loss.

4. Hydrostatic pressure distribution is assumed.



Longitudinal section



Plan

The total energy H and q $\left(= \frac{Q}{b} = \text{discharge per unit width} \right)$ are constant,

$$H = y + z + \frac{q^2}{2gy^2} = E + z = \text{constant}$$

differentiating with respect to x, the distace along the channel

$$\frac{dE}{dx} + \frac{dz}{dx} = 0$$

which may be rewritten as

$$\frac{dE}{dy}\frac{dy}{dx} + \frac{dz}{dx} = 0$$

Substituting and simplifying

$$\frac{dy}{dx}\left(1-F^2\right) + \frac{dz}{dx} = 0 \qquad \left(\because \frac{dE}{dy} = 1-F^2;\ F = \frac{\overline{V}}{\sqrt{gy}}\right)$$

$$E = y + \frac{\overline{V}^2}{2g}$$

$$\frac{dE}{dy} = 1 + \frac{d}{dy}\left(\frac{Q^2}{2gA^2}\right) = 1 + \frac{Q^2}{2g}\left(-2A^{-3}\frac{dA}{dy}\right)$$

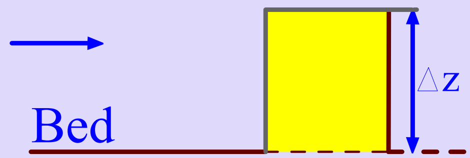
$$\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3}T = 1 - F^2$$

$$\text{(i.e)}\ F^2 = \frac{Q^2 T}{gA^3}$$

It is to be noted that the Froude number F plays a key role in this equation. This equation demonstrates in nutshell from a result from the E-y curve.

If there is an upward step in the channel bed, i.e., if dz/dx is positive, then the product $\left(1-F^2\right)\frac{dy}{dx}$ must be negative and vice versa (see box).

If $\frac{dz}{dx}$ is positive

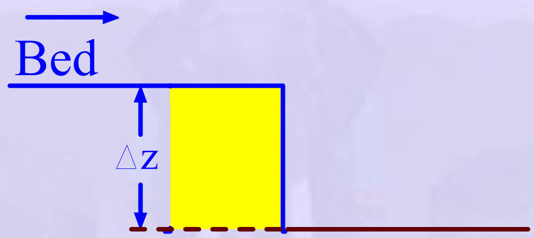


$$\frac{dy}{dx}\left(1-F^2\right) = \text{negative}$$

$F<1$ (Subcritical) $\frac{dy}{dx}$ -ve (depth decreases along x)

$F>1$ (Supercritical) $\frac{dy}{dx}$ +ve (depth increases along x)

If $\frac{dz}{dx}$ is negative



$$\frac{dy}{dx}\left(1-F^2\right) = \text{positive}$$

$F<1$ (Subcritical) $\frac{dy}{dx}$ +ve (depth increases along x)

$F>1$ (Supercritical) $\frac{dy}{dx}$ -ve (depth decreases along x)

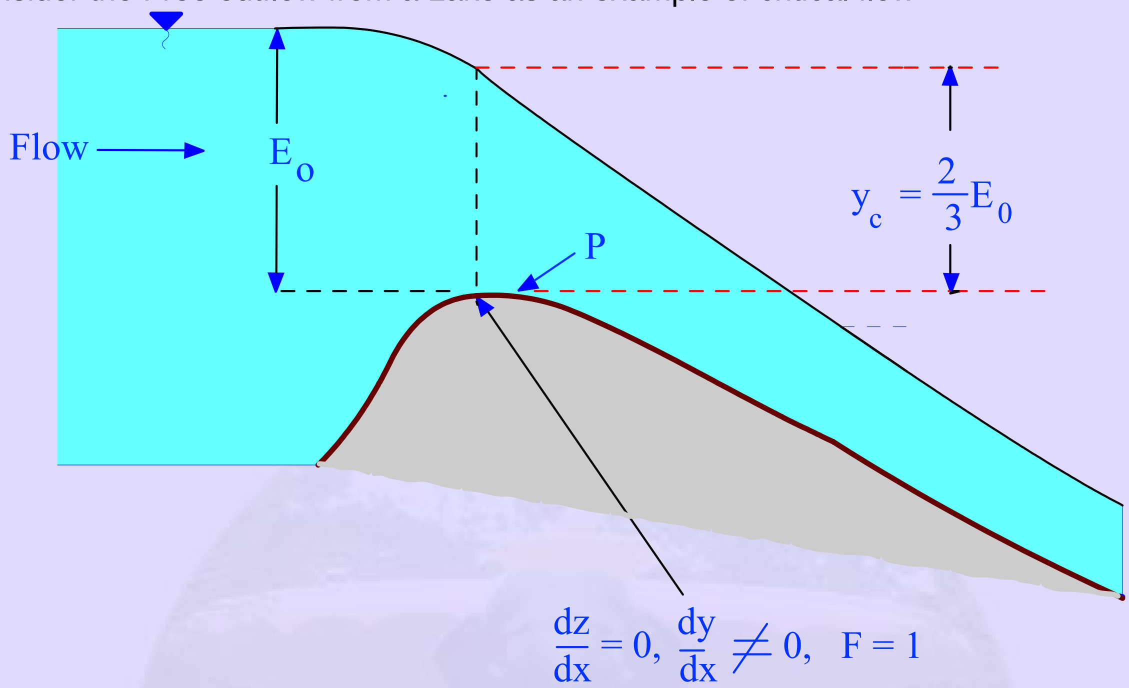
However, if the channel bed is horizontal i.e., $\frac{dz}{dx}=0$,. Then the product $\left(1\text{-}F^2\right)\frac{dy}{dx}$ is then equal to zero. Hence, either $\frac{dy}{dx}=0$ or $F = 1$ (critical flow).

The first situation occurs in the step-transition problem when $\frac{dz}{dx}=0$, $\frac{dy}{dx}=0$ both upstream of the step and over the step, and in both cases $F \neq 1$.

For the second situation, the question is " Can a situation be visualized in which $\frac{dz}{dx}=0$ and $\frac{dy}{dx} \neq 0$?

The answer is yes.

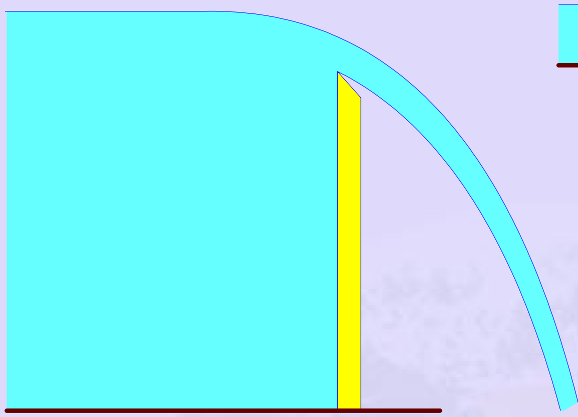
Consider the Free outflow from a Lake as an example of critical flow.



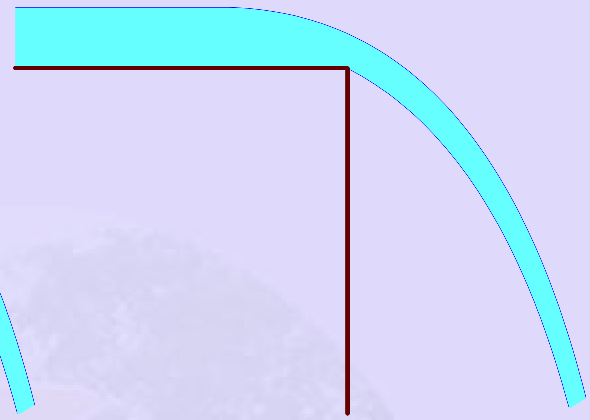
An example of Critical - Free Outflow from a Lake

When water is released from a lake over a short (but smooth) crest such that it flows downstream freely. In other words either a free overfall within a short distance downstream or a steep slope whose bed resistance imposes no effective constraint on the flow.

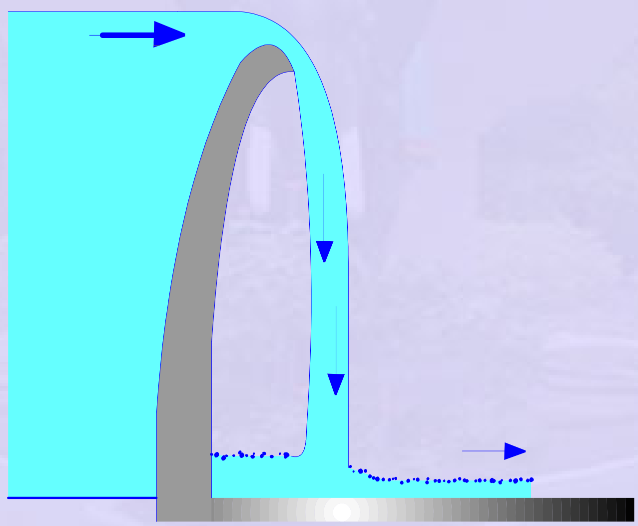
At the crest P, $\frac{dz}{dx} = 0$ the flow is accelerating at this point, resulting in $\frac{dy}{dx} \neq 0$. Then the Froude number must be equal to unity, and hence the flow would be critical. In cases of a sharp- edged (e.g., V notch weir) crest, and a completely free overfall, are considered as pressure distribution would be non hydrostatic; for the reason the curvature will not be large. However, even if the vertical accelerations is large, as near brink of a free overfall, the flow is still can be approximated as the critical condition. Experimental evidence indicates that the flow depth right at the brink of an overfall is approximately $\frac{5}{7}y_c$, (i.e 0.715 y_c) and that $y = y_c$ at a distance upstream from the overall edge of weir of infinite height, the discharge is remarkably close to that obtained by assuming critical flow at the crest, despite the pronounced vertical curvature of the flow. Assuming that the pressure distribution is hydrostatic, it can be concluded that when water is released from a lake without any downstream constraint critical flow occurs at the section of maximum vertical constriction: such a section is therefore a control. Similarly that critical flow occurs at a corresponding horizontal constriction.



Free overfall over a sharp crested weir



Free overfall (drop)

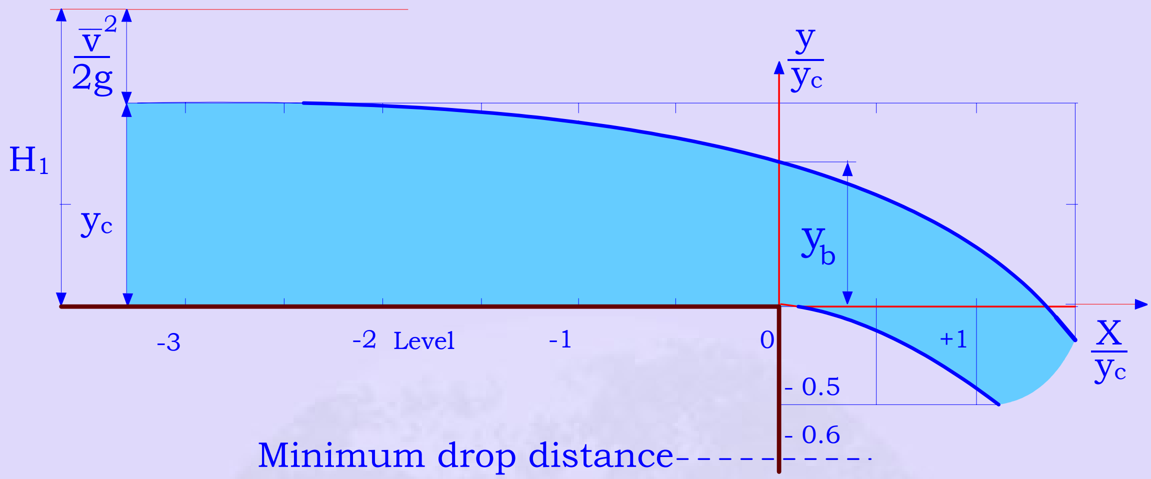


Free Over fall over an arch dam

10.2.1 End Depth or Brink Depth

When the channel terminates abruptly the end weir is known as “The Weir of Zero height". The flow in the end reach of the channel becomes an overfall. Measuring the depth at the end section of the channel, the discharge can be estimated. Rouse first identified this aspect in a horizontal rectangular channel (with sub critical approach flow). The end depth (also called the brink depth) was 0.715 times the critical depth.

When the canal drops suddenly, a free overfall is formed, since flow changes to supercritical flow can be used as a measuring device.



Free overfall profile

The drop distance should be more than 0.6yc. Brink depth y_b will be different at the centre and sides of the canal (which is higher). The roughness of the canal affects the brink depth and hence the bed and sides should be finished smooth.

$$H_o = y + \alpha \frac{q^2}{2gy^2}$$

Differentiating w.r.t 'y' assuming Q to be constant.

$$\frac{dH_o}{dy} = 1 - \alpha \frac{q^2}{gy^3}$$

$\frac{dH_o}{dy} = 0$ if the flow is critical, hence $y_c = \sqrt[3]{\frac{\alpha q^2}{g}}$

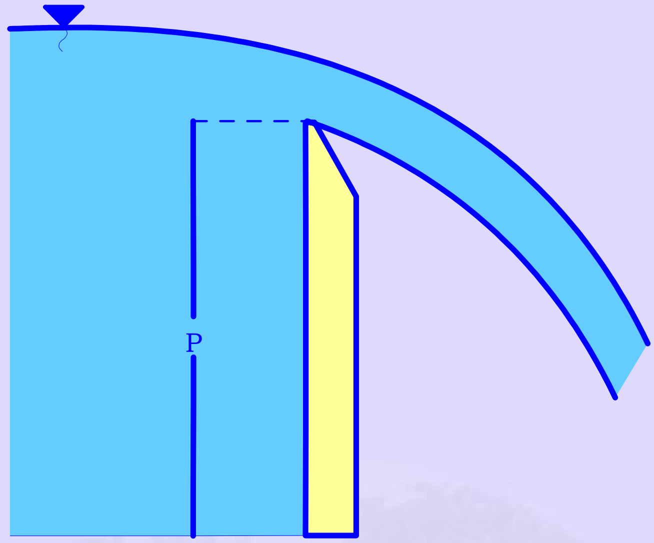
$$\text{If } \alpha = 1\text{, then } Q = b\sqrt{g}\ y_c^{3/2}$$

$$\text{Rouse showed } y_b = 0.715 y_c$$

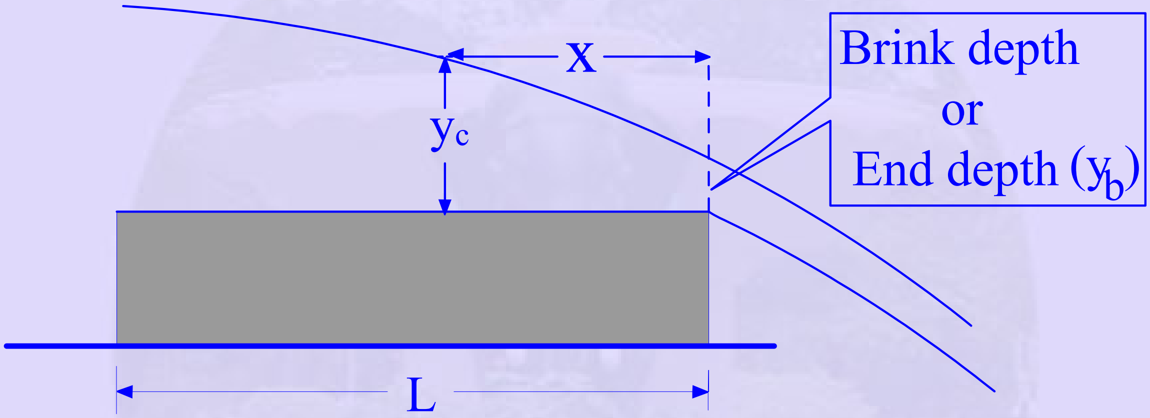
$$\text{Thus } Q = b\sqrt{g}\left[\frac{y_b}{0.715}\right]^{3/2}$$

This derivation is assumed for a free fall with an unconfined nappe. This value is modified as 0.705 when the flow is two dimensional. This results in a error of 2 to 3 % respectively for the above two cases.

The width of the canal should not be less than 3 y_c. This is applicable to canals with slopes upto 0.0025.



Thin weir plate - Free over fall



$$\frac{y_c}{y_b} \approx 1.4 \text{ , } x = 3 \text{ to } 4\, y_c$$

Brink depth

10.2.2 Constriction in bed width

In case of a horizontal channel bed and a variable width b, the energy equation can be written, taking z as a constant but q as a variable function of x as

$$\text{Total energy} = \text{TE} = H = z + y + \frac{V^2}{2g} \qquad (\alpha = 1.0)$$

$$=> H = z + y + \frac{Q^2}{2gA^2} = z + y + \frac{q^2 \cancel{b^2}}{2g \cancel{b^2} y^2}$$

$$H = y + z + \frac{[q(x)]^2}{2gy^2}$$

Differentiating both sides with respect to " x ",

$$\frac{dH}{dx} = \frac{dy}{dx} + \frac{dz}{dx} + \frac{d}{dx}\left\{\frac{[q(x)]^2}{2gy^2}\right\} = 0$$

If $\frac{dH}{dx} = 0$ and $\frac{dz}{dx} = 0$ (No energy loss, Horizontal channel)

$$\frac{dy}{dx} - \frac{q^2}{gy^3}\frac{dy}{dx} + \frac{q}{gy^2}\frac{dq}{dx} = 0$$

and by continuity equation q b = a constant, Q.

Then

$$\frac{dQ}{dx} = 0 = b\frac{dq}{dx} + q\frac{db}{dx} = 0$$

$$b\frac{dq}{dx} = -q\frac{db}{dx}$$

Eliminating $\frac{dq}{dx}$, between above two equations then it may be written as

$$\frac{dy}{dx}\left(1-F^2\right) - \frac{q}{gy^2}\frac{q}{b}\frac{db}{dx} = 0$$

$$\text{i.e., } \frac{dy}{dx}\left(1-F^2\right) - F^2\frac{y}{b}\frac{db}{dx} = 0$$

It can be concluded that critical flow occurs when $\frac{db}{dx}$, i.e., at a section of maximum horizontal constriction. The critical flow will not occur at a section of maximum width, but only at a section of minimum width.

Converging

(i) $\frac{db}{dx} < 0$ F<1 subcritical then $\frac{dy}{dx} < 0$ depth decreases as x increases

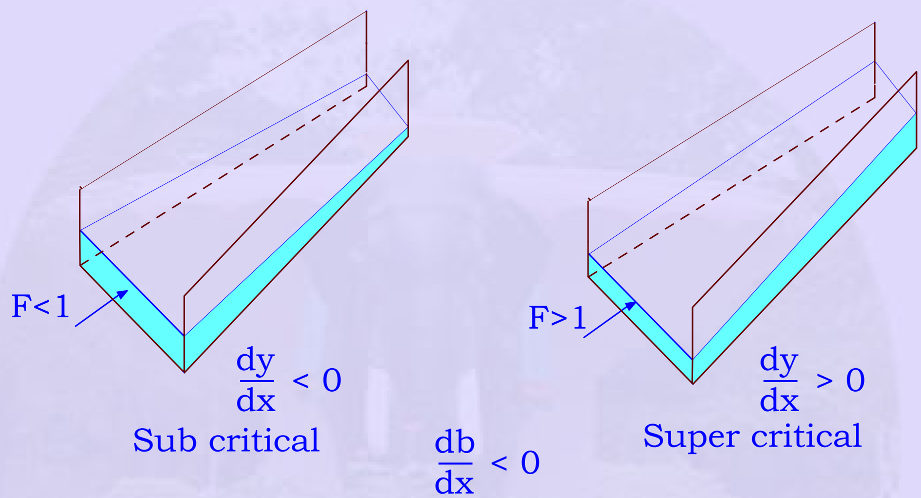
F>1 supercritical then $\frac{dy}{dx} > 0$ depth increases as x increases

Diverging

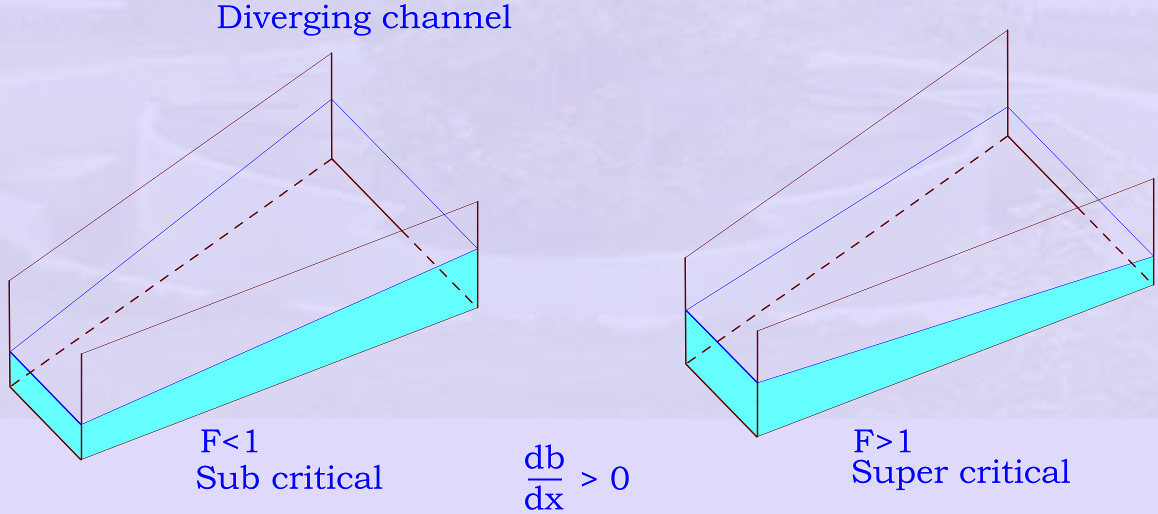
(i) $\frac{db}{dx} > 0$ F<1 subcritical then $\frac{dy}{dx} > 0$ depth increases as x increases

F>1 supercritical then $\frac{dy}{dx} < 0$ depth increases as x increases

Converging channel



Diverging channel



Horizontal constriction

Derive the following equation for a non prismatic channel, assuming no energy loss.

$$\frac{dy}{dx} = \boxed{\frac{S_o + \frac{y_c^3}{by^2}.\frac{db}{dx}}{1-\left(\frac{y_c}{y}\right)^3}}$$

Solution:

Total energy at any section is given by

$$H = z + y + \frac{V^2}{2g} \qquad (\alpha = 1.0)$$

Differentiating wrt "x",

$$\frac{dH}{dx} = \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left\{\frac{V^2}{2g}\right\} \text{------------} > (1)$$

$$\text{But } \frac{dH}{dx} = -S_f$$

$$\text{Similarly } \frac{dz}{dx} = -S_o \qquad \text{But } S_f = 0 \implies \frac{dH}{dx} = 0$$

Substituting in eq: (1),

$$0 = -S_o + \frac{dy}{dx} + \frac{d}{dx}\left\{\frac{V^2}{2g}\right\} \text{---------} > (2)$$

$$0 = -S_o + \frac{dy}{dx} + \frac{d}{dx}\left\{\frac{Q^2}{2gA^2}\right\}$$

Consider a rectangular channel with varying width

$$\frac{d}{dx}\left\{\frac{Q^2}{2gA^2}\right\} = \frac{d}{dx}\left\{\frac{Q^2}{2g\,b^2\,y^2}\right\} = \frac{Q^2}{2g}\frac{d}{dx}\left\{\frac{1}{b^2\,y^2}\right\}$$

$$= \frac{Q^2}{2g}\left\{\frac{-2}{b^3\,y^2}\frac{db}{dx} - \frac{-2}{b^2\,y^3}\frac{dy}{dx}\right\}$$

$$= \frac{Q^2}{gb^3\,y^2}\frac{db}{dx} - \frac{Q^2}{gb^2\,y^3}\frac{dy}{dx}$$

Substituting this expresstion in eq: (2),

$$- S_o + \frac{dy}{dx} - \frac{Q^2}{gb^3\, y^2}\frac{db}{dx} - \frac{Q^2}{gb^2\, y^3}\frac{dy}{dx} = 0$$

$$=> - S_o + \frac{dy}{dx}\left(1 - \frac{q^2}{gy^3}\right) - \frac{q^2}{gb\, y^2}\frac{db}{dx} = 0$$

But $\frac{q^2}{g} = y_c^3$

$$\therefore \quad \frac{dy}{dx} = \boxed{\frac{S_o + \frac{y_c^3}{by^2}\cdot\frac{db}{dx}}{1 - \left(\frac{y_c}{y}\right)^3}}$$

: