19.4 Solution of Manning Equation by Newton Raphson Method

There is no general analytical solution to manning equation for determining the flow depth given the flow rate because the area A and hydraulic radius R may be complicated functions of the depth. Newton Raphson method can be applied iteratively to obtain a numerical solution. Suppose that at iteration k the depth y_k is selected and the flow rate Q_n, is computed using manning formula using the area and hydraulic radius corresponding to y_k. This Q_k is compared with actual flow Q_n; then the objective is to chose y such that the error.

$f(y_k) = Q_k - Q_n$ is within the tolerance limit. The gradient of f with respect to y is

$$\frac{df(y_k)}{dy_k} = \frac{dQ_k}{dy_k}$$

because Q_n is constant. Hence, assuming manning roughness is constant,

$$\left(\frac{df}{dy}\right)_k = \left(\frac{1}{n} S_o^{1/2} A_k R_k^{2/3}\right)$$

$$= \frac{1}{n} S_o^{1/2} \left(\frac{2A R^{-\frac{1}{3}}}{3}\frac{dR}{dy} + R^{\frac{2}{3}}\frac{dA}{dy}\right)_k$$

$$= \frac{1}{n} S_o^{1/2} A_k R_k^{2/3} \left(\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy}\right)_k$$

$$\left(\frac{df}{dy}\right)_k = Q_k \left(\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy}\right)_k$$

in which the subscript k out side the bracket indicates that the quantities in the bracket computed for $y = y_k$.

In Newton's method,

given a choice of y_k, y_{k+1} is chosen to satisfy

$$\left(\frac{df}{dy}\right)_k = \frac{0 - f(y)_k}{y_k + y_{k+1}}$$

This y_{k+1} is the value of y_k,

$$y_{k+1} = y_k - \frac{f(y_k)}{(df/dy)_k}$$

Which is the fundamental equation of the Newton's method. Iterations are continued until there is no significant change in y_n; this will happen when the error is nearly zero or an acceptable prescribed tolerance.

Thus for manning equation it may be written as

$$y_{k+1} = y_k - \frac{1 - Q/Q_k}{\left(\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy}\right)_k}$$

For rectangular channel $A = b_o y$ and $R = b_o y / (b_o + 2y)$ where b_o is the channel width; The quantity in denominator can be for rectangular channel

$$\frac{d}{dy}(\mathbf{R}) = \frac{d}{dy}\left(\mathbf{A}/\mathbf{P}\right)$$

$$= \frac{1}{P}\frac{dA}{dy} - \frac{A}{P^2}\frac{dP}{dy}$$

$$= \left[\frac{T}{P} - \frac{R}{P}\frac{dP}{dy}\right]$$

consider

$$\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy}$$

$$\frac{2}{3}\frac{P}{A}\left[\frac{T}{P} - \frac{R}{P}\frac{dP}{dy}\right] + \frac{T}{A}$$

$$\frac{2}{3}\frac{1}{A}\left[T - R\frac{dP}{dy}\right] + \frac{T}{A}$$

$$\frac{2}{3}\frac{T}{A} - \frac{2}{3}\frac{R}{A}\frac{dP}{dy} + \frac{T}{A}$$

$$\left[\frac{5}{3}\frac{T}{A} - \frac{2}{3}\frac{1}{P}\frac{dP}{dy}\right]$$

For rectangular channel

$$\frac{5}{3}\frac{b_o}{b_o y} - \frac{2}{3}\frac{1}{(b_o + 2y)}2$$

$$\frac{5}{3}\frac{1}{y} - \frac{4}{3}\frac{1}{(b_o + 2y)}$$

$$\frac{5(b_o + 2y) - 12y}{3y(b_o + 2y)} = \frac{5b_o + 10y - 4y}{3y(b_o + 2y)}$$

$$= \frac{5b_o + 6y}{3y(b_o + 2y)}$$

$$y_{k+1} = y_k - \frac{1 - Q/Q_k}{\left(\frac{5\,b_o + 6\,y_k}{3y_k(b_o + 2\,y_k)}\right)}$$

Similarly the channel shape function $\left[(2/3R)(dR/dy) + (1/A)(dA/dy)\right]$ for other cross sections can be derived.

Trapezoided Channel $\dfrac{\left(b_o+2my\right)+6y\sqrt{\left(1+m^2\right)}+4my^2\sqrt{\left(1+m^2\right)}}{3y\left(b_o+my\right)\left(b_o+2y\sqrt{\left(1+m^2\right)}\right)}$

Triangular Channel $\dfrac{8}{3y}$

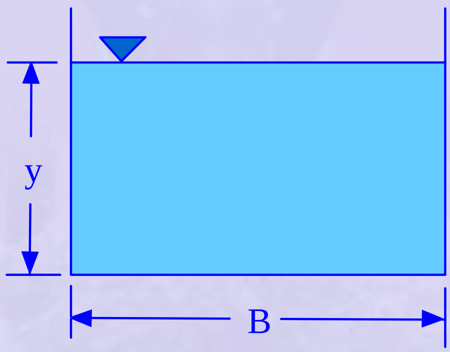
Circular Conduit $\dfrac{4\left(2\sin\theta+3\theta-5\theta\cos\theta\right)}{3d_o\left(\theta\right)\left(\theta-\sin\theta\right)\sin\left(\frac{\theta}{2}\right)}$

in which

$$\theta=2\cos^{-1}\left(1-\frac{2y}{d_o}\right)$$

Example:

Compute the flow depth in a 0.6 m wide rectangular channel having n= 0.015, S_0 = 0.025, and Q = 0.25 m^3s^{-1}.



Solution:

Let wide $b_o=0.6m$

Manning coefficient $n=0.015$

bed slope $S_o=0.025$

discharge $Q=0.25m^3s^{-1}$

normal depth $y=?$

Hyraulic mean radius $R=\dfrac{A}{p}=\dfrac{b_o}{b_o+2y}$

$$Q=\frac{1}{n}AR^{\frac{2}{3}}S_o^{\frac{1}{2}}$$

$$Q=\frac{1}{n}b_oy\left(\frac{by_k}{b+2y_k}\right)^{\frac{2}{3}}S_o^{\frac{1}{2}}$$

$$Q = \frac{1}{n}\left[\frac{(by_k)^{\frac{5}{3}}}{(b+2y_k)^{\frac{2}{3}}}\right]S_o^{\frac{1}{2}}$$

$$Q_k = \frac{1}{0.015}\times\left[\frac{(0.6\times y_k)^{\frac{5}{3}}}{(0.6+2y_k)^{\frac{2}{3}}}\right]\times(0.025)^{\frac{1}{2}}$$

$$Q_k = 10.5409 * \frac{0.6^{5/3}\, y_k^{5/3}}{(0.6+2y_k)^{2/3}} = \frac{4.4993 y_k^{5/3}}{(0.6+2y_k)^{2/3}} \qquad (1)$$

$$\text{Shape function} = \frac{5b_o + 6y_k}{3y_k(b+2y_k)}$$

$$= \frac{5(0.6)+6y_k}{3y_k(0.6+2y_k)} = \frac{3+6y_k}{3y_k(0.6+2y_k)} = \frac{1+2y_k}{y_k(0.6+2y_k)}$$

$$y_{k+1} = y_k - \frac{\left(1-\frac{0.25}{Q_k}\right)y_k(0.6+2y_k)}{(1+2y_k)} \qquad (2)$$

Iteration (k)	1	2	3
$y_k(m)$	0.100	0.1815	0.1727
Q(m^3s^{-1})	0.1125	0.2684	0.2488

$$\text{Froude number F} = \frac{V}{\sqrt{gy}} = \frac{Q/A}{\sqrt{gy}}$$

$$F = \frac{0.2488/(0.6*0.1727)}{\sqrt{(9.81*0.1727)}} = 1.844$$

$\therefore$ super critical flow