22.4 Most efficient Hydraulic section

For a given cross section determine the hydraulic section.

Hydraulically Best section (Hydraulically Efficient Section)

1. Rectangular Channel:

$$P = b + 2y$$

$$b = \frac{A}{y}$$

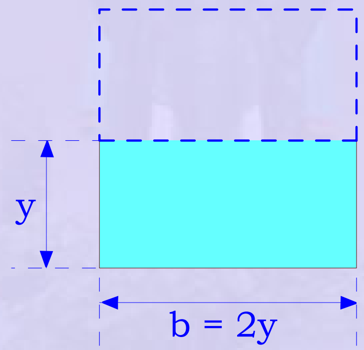
$$A = by$$

$$P = \frac{A}{y} + 2y$$

$$\frac{dp}{dy} = -Ay^{-2} + 2 = 0,\ A = 2y^2$$

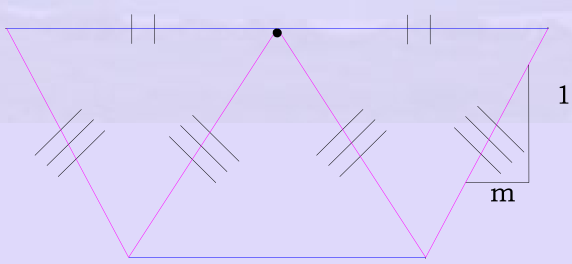
$$\therefore by = 2y^2$$

or $b = 2y$



Hydraulically efficient rectangular channel is half of a square.

2. Trapezoidal Section:



$$P = b + 2y\sqrt{1+m^2}$$

$A = (b + my)\,y$ or $b = \frac{A}{y} = -my$

$$P = \frac{A}{y} - my + 2y\sqrt{1+m^2}$$ (For a given area of flow)

Differentiate with reference to y assuming A and m to be constant.

$$\frac{dp}{dy} = -Ay^{-2} - m + 2\sqrt{1+m^2} = 0$$

Substituting for area, the above equation can be rewritten as

$$-y^{-2}(b+my)y - m + 2\sqrt{1+m^2} = 0$$

$$+\frac{b+my+my}{y} = +2\sqrt{1+m^2} = 0$$

$$\boxed{\frac{b+2my}{2} = y\sqrt{1+m^2}}$$

Half the top width = side slope distance (for given side slope)

$$b = 2y\sqrt{1+m} - 2my = 2y\left[\sqrt{1+m^2} - m\right]$$

Substitute this value of b into the equation A and P and simplifying

$$P = 2y\left[2\sqrt{1+m^2} - m\right]$$

$$A = y^2\left[2\sqrt{1+m^2} - m\right]$$

$$y = \left[\frac{A}{2\sqrt{1+m^2} - m}\right]^{0.5}$$

Substitute the value of y into P

$$P = 2\frac{A^{1/2}}{\left(2\sqrt{1+m^2} - m\right)^{0.5}}\left(2\sqrt{1+m^2} - m\right)$$

$$P = 2\sqrt{A\left(\left(2\sqrt{1+m^2} - m\right)\right)}$$

which is the m value that makes P least?

D.w.r to m and equate it to zero

$$\frac{dP}{dm} = 0$$

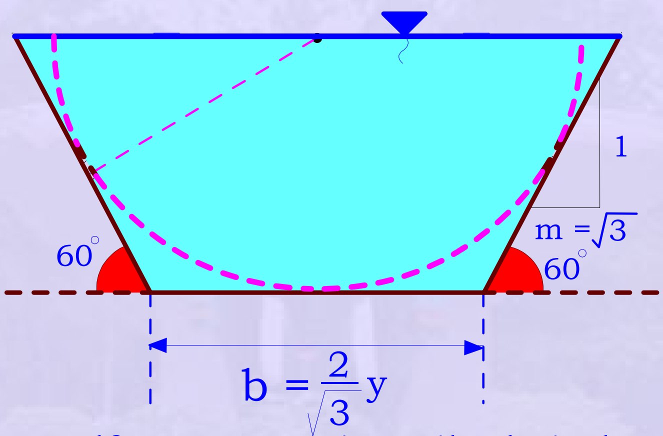
$$m = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$tan\,\theta = \frac{y}{my} = \frac{1}{m} = \frac{\sqrt{3}}{3}$$

$$\therefore \theta = 60^\circ$$

$$\therefore b = \frac{2}{\sqrt{3}}\, y$$

This means section is a half hexagon. If a semi circle is drawn with radius equal to depth y then sides of this section are tangent to the circle.



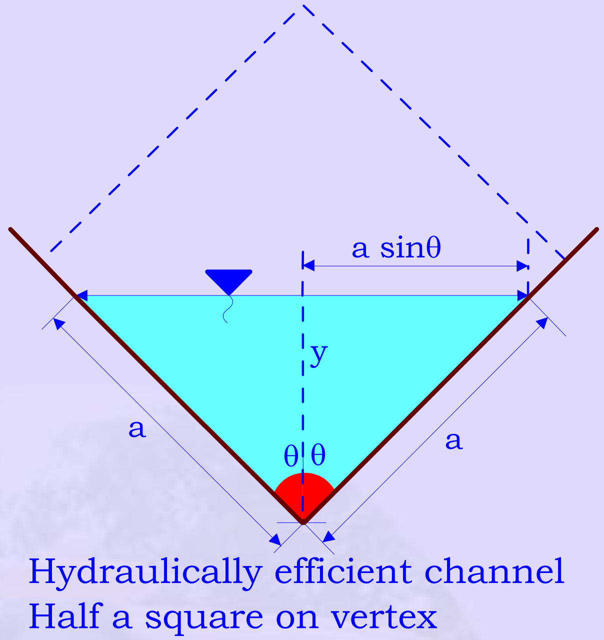
Half Hexagon - inscribed circle of radius equal to depth is tangential as shown in figure

Triangular Section: If θ is half angle

$$Area = ay\sin\theta \left[\because \frac{2a\sin\theta}{2} y = ay\sin\theta\right]$$

$$R = \frac{ay\sin\theta}{2a} = \frac{a\cos\theta\sin\theta}{2} = \frac{a}{4}\sin 2\theta$$

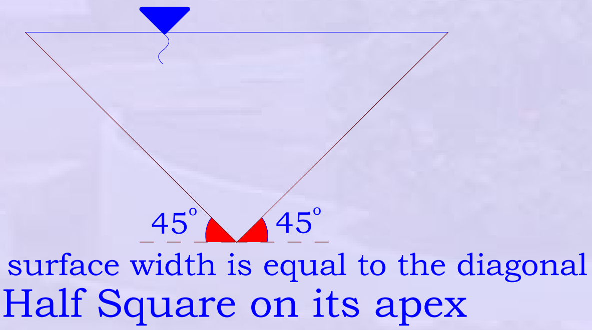
$$\left[\because \frac{y}{a} = \cos\theta\right]$$



R should be max.

$$\frac{dR}{d\theta} = 0, \quad \frac{2a}{4}\cos 2\theta = 0$$

$$\therefore \theta = 45^\circ$$



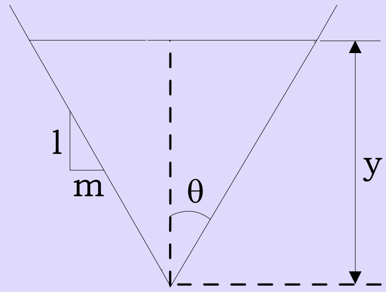
It is a half square resting on its apex and maximum width is equal to diagonal.

Alternative derivation for Triangular Section:

$$A = y^2\tan\theta$$

$$y = \sqrt{A/\tan\theta}$$

$$P = 2y\sqrt{1+m^2}$$



$$= 2y \sec\theta$$

$$= 2\sqrt{\frac{A}{\tan\theta}} \sec\theta$$

$$\frac{dp}{d\theta} = \frac{d}{d\theta}\left\{2\sqrt{\frac{A}{\tan\theta}} \sec\theta\right\}$$

$$\frac{dp}{d\theta} = 2\sqrt{A}\left[\frac{\sec\theta \tan\theta}{\sqrt{\tan\theta}} - \frac{\sec^3\theta}{2(\tan\theta)^{3/2}}\right] = 0$$

$$\sec\theta \tan\theta - \frac{\sec^3\theta}{2(\tan\theta)^3} = 0$$

$$2\sec\theta \tan\theta - \sec^3\theta = 0$$

$\therefore$ Solve for θ

$\therefore \ \theta = 45^\circ$

Hydrostatic Caternary (Linteria)

Equation for hydrostatic catenary is given by

$$x_1 = \frac{y}{2k}\left[\left(1-\frac{3}{4}k^2-\frac{15}{864}k^4\right)\phi+\left(\frac{3}{8}k^2+\frac{5}{32}k^4\right)\sin 2\phi-\frac{5}{256}k^4\sin 4\phi\right]$$

$$y_1 = y\cos\phi$$

x_1, y_1 are measured from mid point of the surface

$k=\sin\frac{\theta}{2}$; θ_o = slope angle at the point $x_1 y_1$. θ_o varies from 0 at the bottom of the curve to θ_o at the ends.

$$\phi=\sin^{-1}\frac{\left[\sin\frac{\phi}{2}\right]}{k},\ \theta \text{ is slope at any point } (x, y)$$

For the hydraulically efficient channel

θ_o = 35° 37' 7" , y= 3.5 m. Find A, R, D, Z at full depth.

Also plot the cross section of the channel

solution: $\frac{\theta_o}{2}=\frac{35°\ 37'\ 7"}{2}=17°48'33.5"$

$$k=\sin\frac{\theta_o}{2}=0.30585$$

$$\sin\phi=\frac{\sin\frac{\phi}{2}}{k} \quad \text{or } k=\frac{\sin\frac{\phi}{2}}{\sin\phi}$$

$$k=\frac{\sin\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}=\frac{1}{2\cos\frac{\phi}{2}}$$

or $$\cos\frac{\phi}{2}=\frac{1}{2k}$$

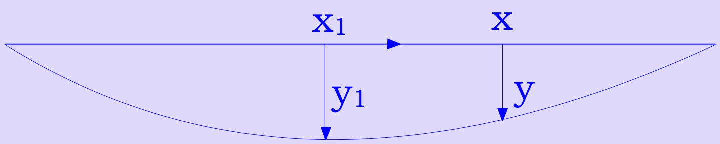
$y_1 = y\cos\phi \qquad \phi=90^\circ \qquad \cos\frac{\phi}{2}=\cos 45 \qquad \therefore k=0.707$

$A=17.0992\ m^2 \qquad P=10.443\ m \qquad R=\frac{A}{P}=1.6374\ m$

$T=6.7114\ m \qquad D=2.5478\ m \qquad Z=A\sqrt{D}=22.293\ m^{5/2}$

y_1 (m)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
ϕ (in deg)	90	81.78	73.39	64.62	55.15	44.4	31.0	0
k	0.7071	0.6614	0.6235	0.5916	0.564	0.5400	0.51887	0.5

Exercise: Plot the graph using the above data

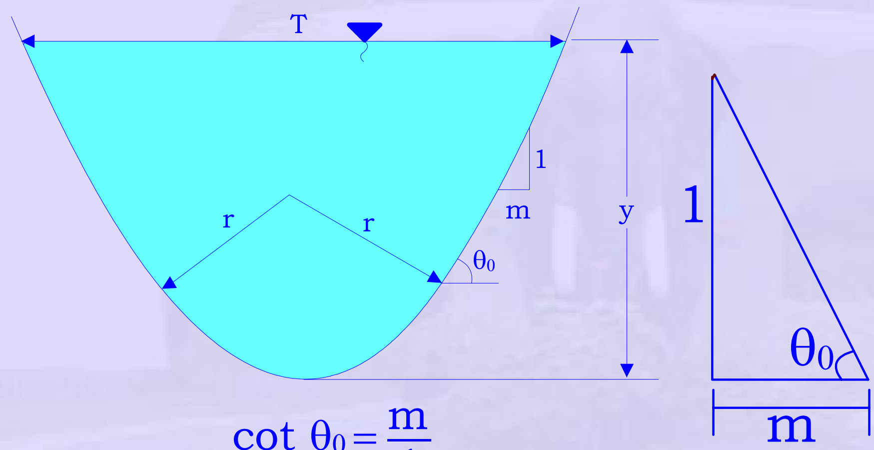


$A = 17.0992 \text{ m}^2$ $\quad$ $P = 10.443 \text{ m}$ $\quad$ $R = \frac{A}{P} = 1.6374 \text{ m}$

$T = 6.7114 \text{ m}$ $\quad$ $D = 2.5478 \text{ m}$ $\quad$ $Z = A\sqrt{D} = 22.293 \text{ m}^{5/2}$

Flexible Sheet: Filled with water upto rim, and held firmly at the top ends without any effect of fixation on shape. Shape assumed under self weight of water is called Hydrostatic Catenary.

Rounded bottom triangular section



$$\cot \theta_0 = \frac{m}{1}$$

$$\therefore \theta_0 = \cot^{-1} m$$

$$A = y^2\left(m^2 \cot^{-1} m\right)$$

$$P = 2y\left(m + \cot^{-1} m\right)$$

$$R = \frac{A}{P} = \frac{r}{2} = \frac{y}{2}$$

$$T = 2r\sqrt{\left(1+m^2\right)}$$

Hydraulically efficient sections could be derived using Lagrange Multiplier approach.