27. Integration of Differential Equation

Computation of water surface profile can be done by numerically solving the non-linear ordinary differential equation (Eq. 23.7). Three methods: (i) Euler method, (ii) Improved Euler method, and (iii) Fourth - order Runge - Kutta method, are presented here.

27.1 Euler Method

Referring to Fig. 27.1, say flow depth Y_i at a distance X_i from the reference point is known. We also know the flow rate, Q, the roughness coefficient, n, the channel slope, S_0 and the channel cross sectional shape parameters. We want to determine the flow depth, Y_{i+1} at a distance X_{i+1}.

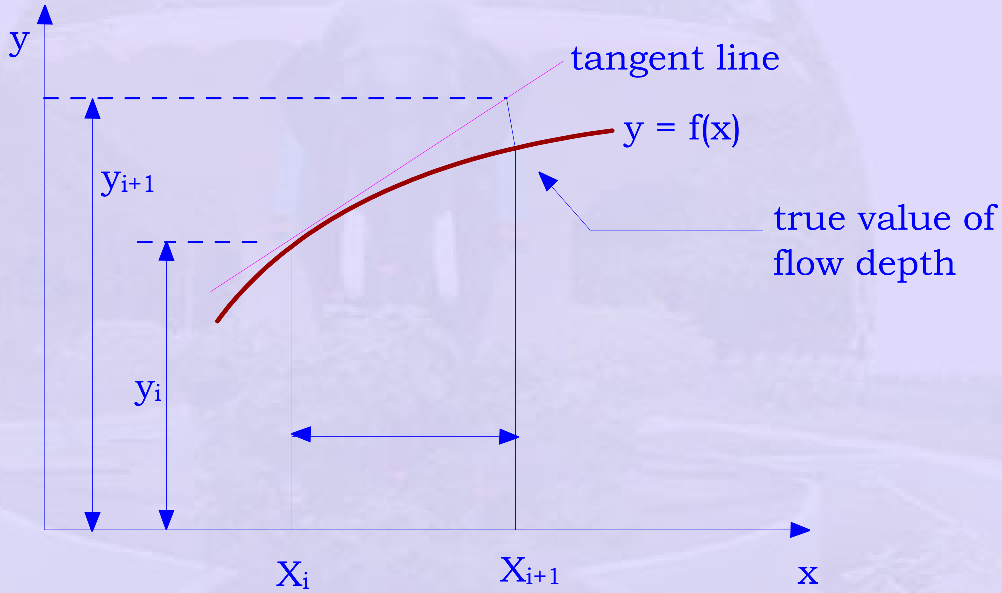

Fig. 27.1: Schematic representation of the Euler method

Rate of variation of flow depth, y with a distance, x can be evaluated as follows:

$$\frac{dy}{dx} = f(x, y) \qquad (27.1)$$

Considering the point, i

$$\left(\frac{dy}{dx}\right)_i = f(x_i, y_i) \qquad (27.2)$$

From the governing equation (Eq. 23.7),

$$f\left(x_i, y_i\right) = \frac{S_0 - S_{f\,i}}{1 - \frac{\alpha Q^2 T_i}{g\left(A_i\right)^3}} \qquad (27.3)$$

In Eq. (27.3), subscript i indicates the value of variable evaluated at distance X_i. $S_{f\,i}, T_i$ and A_i are dependent on the flow depth Y_i , and therefore, they can be determined explicitly. Thus at any location i, $f\left(x_i,\ y_i\right)$ can be evaluated. This is nothing but the slope of the line drawn tangent to the curve y = f(x) at $x = x_i$ (Fig 27.1). If we assume that this value does not change in the interval from x_i to x_{i+1}, flow depth at $x = x_{i+1}$ can be determined as follows.

$$y_{i+1} = y_i + f\left(x_i, y_i\right)\ \Delta x \qquad (27.4)$$

The above method is known as Euler's Method.

Once Y_{i+1} is known, we can determine Y_{i+2} at location x_{i+2} by repeating the above procedure. Referring to Fig. 27.1, it can be seen that there is a difference between the estimated value of flow depth Y_{i+1} and its true value. Taylor series expansion would show that Euler's method is only first - order accurate. This error in the computation of flow depth at each step may get magnified as the value of x increases, and therefore, this method is usually unstable. Very small values of Δx may be required to obtain satisfactory results.

Example

A wide rectangular channel having a bottom slope of 0.001 is carrying a flow of 3 m^3/s/m. Flow depth at a particular location is 2.0 m. Determine the flow depth at a distance 500 m downstream of this point. Manning n for the channel is 0.012. Use Euler's method.

Solution

- $y_i = 2.0 \text{ m}$
- $T_i = 1.0 \text{ m (unit width)}$
- $A_i = y_i * 1 = 2.0 \text{ m}^2$
- $q = 3 \text{ m}^2/\text{s}$
- $\alpha_i = 1.0 \text{ (assumed)}$
- $S_{f_i} = \dfrac{q^2 n^2}{(y_i)^{10/3}} = 1.2858*10^{-4}$
- $f(x_i, y_i) = \dfrac{0.001 - 1.2858 * 10^{-4}}{1 - \dfrac{(3)^2}{9.81*(2)^3}}$

$$= 9.843 * 10^{-4}$$

- $y_{i+1} = y_i + f(x_i, y_i)\Delta x$

$$= 2 + 9.843*10^{-4}*500$$

$$= 2.492 \text{ m}$$

Flow depth at 500 m downstream = 2.492 m