27.3 Fourth - Order - Runge - Kutta Method

In the fourth order Runge - Kutta method, the slope of the water surface profile, f(x,y) is determined as a weighted mean of four slopes as given below.

$$\begin{aligned} s_1 &= f\left(x_i,\ y_i\right) \\ s_2 &= f\left(x_i + \frac{\Delta x}{2},\ y_i + \frac{1}{2} s_1\ \Delta x\right) \\ s_3 &= f\left(x_i + \frac{\Delta x}{2},\ y_i + \frac{1}{2} s_2\ \Delta x\right) \qquad (27.9) \\ s_4 &= f\left[x_i + \Delta x,\ y_i + s_3\ \Delta x\right] \end{aligned}$$

and

$$y_{i+1} = y_i + \frac{1}{6}\left[s_1 + 2s_2 + 2s_3 + s_4\right]\Delta x \qquad (27.10)$$

This method is fourth-order accurate.

Example 27.3.1

A rectangular channel of 5.0 m width carries a discharge of 10 m^3/s. The channel slope is 0.0001 and the Manning's n = 0.018. Flow depth at a particular section in this channel is 2.5 m. Determine the flow depth at a distance of 1000 m downstream of this section.

Solution

q = unit discharge = 10 / 5 = 2 m^2/s

critical depth, $y_c = \left(\frac{q^2}{g}\right)^{1/3} = 0.742\ m$

Normal depth, y_n

$$q = \frac{1}{n} y_n \left(\frac{5y_n}{5 + 2y_n}\right)^{2/3} \sqrt{0.0001}$$

$$\text{or } y_n \left(\frac{5y_n}{5 + 2y_n}\right)^{2/3} = 3.6$$

Solving by trial and error,

y_n = 2.945 m

In this case, $y_n > y_c \Rightarrow$ MILD SLOPE

$y > y_c$ and $y < y_c \Rightarrow M_2$ Profile

Thus the water surface profile is of M2 type. Therefore, the flow depth decreases in the downstream direction.

$$s = \frac{dy}{dx} = \text{Slope} = \frac{s_0 - s_f}{1 - \dfrac{q^2}{gy^3}}$$

$$S_f = \frac{q^2 n^2}{y^2 R^{4/3}}$$

Calculations for slope s_1 (based on $y = y_i = 2.5$ m)

$q = 2\ m^2/s$

$s_0 = 0.0001$

n = 0.018

y = 2.5 m; A = 2.5 * 5 = 12.5 m^2

P=5+2*2.5=10 m; R = 1.25 m

$$S_f = \frac{q^2 n^2}{y^2 R^{4/3}} = 1.54 * 10^{-4}$$

$$s_1 = \frac{0.0001 - 1.54 * 10^{-4}}{1 - \dfrac{q^2}{gy^3}} = -5.544 * 10^{-5}$$

Calculations for slope s_2

Depth for slope s_2 is given by $y_i = s_1 \dfrac{\Delta x}{2} = 2.472\,\text{m}$

y = 2.472 m; A = 12.36 m^2

P = 9.944 m; R = 1.243 m
Based on this; $s_2 = -6.032*10^{-5}$

Calculations for slope s_3

Depth for slope s_3 is given by $y_i + s_2 \dfrac{\Delta x}{2}$ = 2.470 m

y = 2.470 m; A = 12.35 m^2

P = 9.94 m; R = 1.242 m

Based on this ; $s_3 = -6.076 * 10^{-5}$

Calculations for slope s_4

Depth for slope s_4 is given by $y_i + s_3 \Delta x$ = 2.439 m

y = 2.439 m; A = 12.195 m^2

P = 9.878 m; R = 1.235 m

Based on this ; s_4 = - 6.6286 * 10^{-5}

$$y_{i+1} = y_i + \frac{\Delta x}{6}\left[s_1 + 2\left(s_2 + s_3\right) + s_4\right]$$

$$y_i = 2.5 \text{ m}$$

$$\Delta x = 1000 \text{ m}$$

y_{i+1} = Flow depth at 1000 m distance = 2.439 m

Let us consider the solution for the above problem using Euler's method

$y_{i+1} = y_i + s_1 \Delta x = 2.5\text{-}1000*5.544*10^{-5} = 2.445$ m

Thus there is an error of 6 mm if Euler's method is used instead of Runge – Kutta method. The resulting error is significant in case the flow depth is close to the critical depth. Consider the solution to the problem in Example 27.3.1 using the Euler's method.

$y \approx 0.78$ m

[A slightly higher value is taken so that the singularity in GVF equation is avoided.

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F^2}$$

$\frac{dy}{dx}$ becomes infinity when F tends to one or when flow approaches Critical Conditions]

A = 0.78 * 6 = 4.68 m^2

P = 7.56 m

R = 0.619 m

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - \frac{q^2}{gy^3}} = \frac{-3.937*10^{-3}}{0.1408} = -0.028$$

$$y_2 = y_3 - 250.0\left[-0.028\right] = 7.77 \text{ m}$$

This is obviously a wrong answer since flow depth in this case cannot exceed the normal depth value of 2.8 m. Therefore, one has to watch out for numerical errors while applying these schemes for GVF computation. These numerical errors can be reduced by taking small values of Δx and by using higher-order methods such as Fourth-order Runge-Kutta method.