39.4 Celerity

The wave Celerity, C is defined as the relative velocity of a wave with respect to the fluid in which it is travelling. The absolute wave velocity, V_ω is the velocity of the wave with respect to a fixed reference frame. The wave moves either in the direction of flow (downstream direction) or in the opposite direction (upstream direction), for a one dimensional flow. For such a case,

$$V_\omega = V \pm C \qquad (39.26)$$

The positive sign is used for wave travelling in the downstream direction and the negative sign is used for wave travelling in the upstream direction.

For gradually varied unsteady flow, the equation for the wave celerity can be obtained from the governing partial differential equations (39.7) and (39.22). These equations can be written in the vector form as

$$\frac{\partial}{\partial t}\{F\} + [J]\frac{\partial}{\partial x}\{F\} = \{S\} \qquad (39.27)$$

in which $\{F\} = \begin{Bmatrix} Y \\ V \end{Bmatrix}$

$$[J] = \begin{bmatrix} V & A/T \\ g & V \end{bmatrix}$$

$$and \quad \{S\} = \begin{Bmatrix} 0 \\ g(S_0 - S_f) \end{Bmatrix} \qquad (39.28)$$

Matrix $[J]$ is termed as the Jacobian of the system of equations. Eigen values of this matrix determine the type of partial differential equations. Equations are classified as hyperbolic equations if the eigen values are real and distinct. They are classified as parabolic if the eigen values are real and equal. They are classified as elliptic if the eigen values are imaginary. For hyperbolic systems, eigen values also represent the absolute wave velocity. The Eigen values for the present system are solved from the following equation.

$$\begin{vmatrix} V-\lambda & A/T \\ g & V-\lambda \end{vmatrix} = 0 \qquad (39.29)$$

In which λ gives the eigen values. The two eigen values λ_1 and λ_2 are given by

$$\lambda_1 = V + \sqrt{\frac{gA}{T}}$$
$$\lambda_2 = V - \sqrt{\frac{gA}{T}} \qquad (39.30)$$

Here, λ_1= absolute wave velocity in the downstream direction.

λ_2 = absolute wave velocity in the downstream direction.

Therefore, the celerity of the wave is given by

$$C = \sqrt{\frac{gA}{T}} \qquad (39.31)$$

It can be seen from Eq. (39.30) that the wave velocity for the downstream direction is positive while the wave velocity for the upstream direction is negative, when the flow is subcritical $\left[V < \sqrt{gA/T}\right]$. On the other hand, when the flow is supercritical $\left[V > \sqrt{gA/T}\right]$, both λ_1 and λ_2 are positive. This indicates that the waves travel only in the downstream direction when the flow is supercritical. That is why it is often said that supercritical flow knows only what happens on the upstream side or it has an upstream control.

The fact that the eigen values are real and distinct shows that the unsteady flow equations for open-channels constitute a system of non-linear hyperbolic equations. Typically, hyperbolic equations represents the propagation of waves in different media. Governing equations for water hammer in pipes, governing equations for transient gas flows etc., are also represented by this type of equations. As discussed in later sections, Method of characteristics can be used for the solution of these equations.

Celerity given by Eq. (39.31) is valid only when the amplitude of the wave is small. It is not valid for finite amplitude waves such as those created by a dambreak or a fast operation of a sluice gate. In such cases, celerity can be obtained from Eqs. (38.10) {for a general cross-section} or Eq. (38.12) {for a rectangular cross section}. These are

$$C = \sqrt{\frac{gA_2\left(A_2\bar{y}_2 - A_1\bar{y}_1\right)}{A_1\left(A_2 - A_1\right)}} \qquad (39.32)$$

or

$$C = \sqrt{\frac{gy_2\left(y_2 + y_1\right)}{2y_1}} \qquad (39.33)$$

in which, A = Cross sectional area, y = flow depth, $\bar{y}$ = depth to the centroid of the cross sectional area, and g = acceleration due to gravity. Subscript 1 indicates the flow conditions ahead of the wave and subscript 2 indicates the flow conditions behind the wave. For small amplitude waves in a wide rectangular channel, $y_2 \approx y_1$ and Eq. (39.33) reduces to

$$C = \sqrt{gy} \qquad (39.34)$$

which is same as the Eq. (39.31).