6.2 Energy, Momentum coefficients for different velocity distributions

Rehbock obtained

1) For Linear Velocity Distribution

$\alpha = 1 + \varepsilon^2$

$\beta = 1 + \frac{\varepsilon^2}{3}$, **in which** $\varepsilon = \left\{\frac{\mathbf{V_{max}}}{\mathbf{\overline{V}}} - 1\right\}$

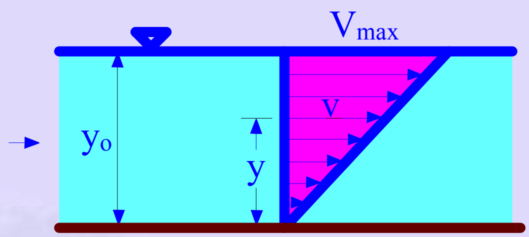
Given: $\alpha = 1 + \varepsilon^2$

=> $\varepsilon^2 = (\alpha - 1)$

Substitution for " ε^2 **" in the expression for "** β **",**

$\beta = 1 + \frac{\alpha - 1}{3} = \frac{3 + \alpha - 1}{3} = \frac{\alpha + 2}{3}$

$\boxed{\beta = \frac{\alpha + 2}{3}}$ **(Linear relation)**

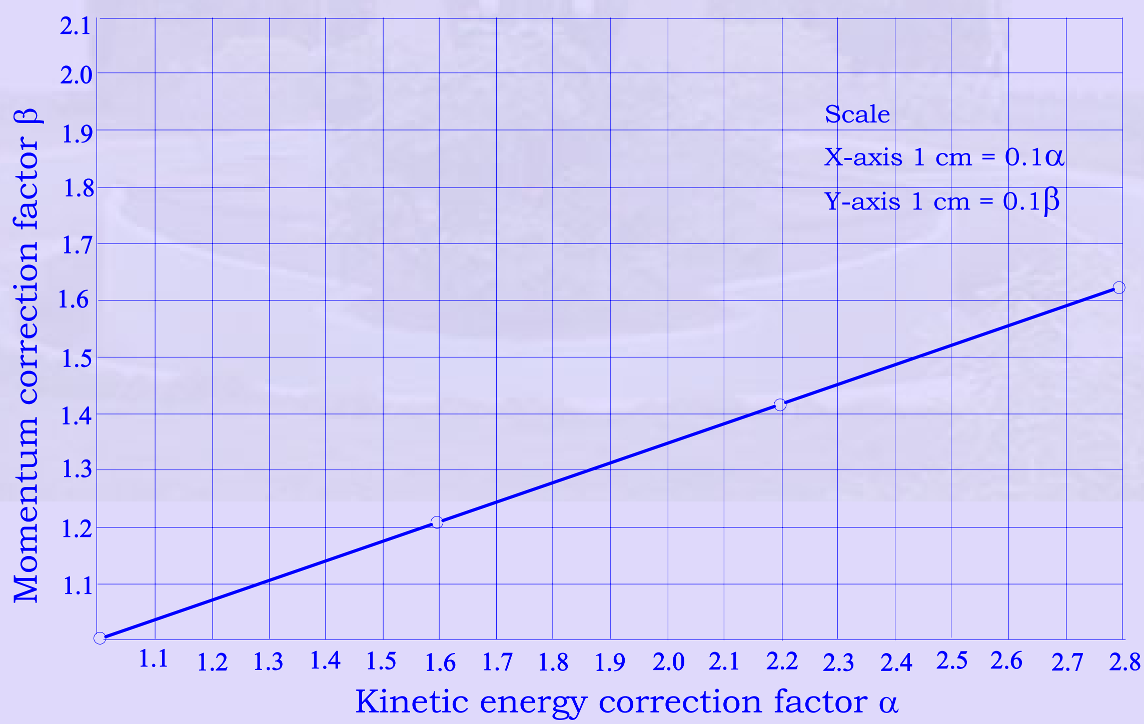


Linear velocity distribution

$\frac{v}{V_{max}} = \frac{y}{y_o}$

α	1	1.6	2.2	2.8
β	1	1.2	1.4	1.6

The plot is shown below



(2) He obtained for Logarithmic Velocity Distribution the following equations.

Kinetic Energy correction factor, $\alpha = 1 + 3\varepsilon^2 - 2\varepsilon^3$

Momentum correction factor, $\beta = 1 + \varepsilon^2$

in which $\varepsilon = \dfrac{2.5\, v_*}{\overline{V}}$

Given : $\beta = 1 + \varepsilon^2$

$=> \varepsilon^2 = (\beta - 1)$

$=> \varepsilon = \sqrt{(\beta - 1)}$

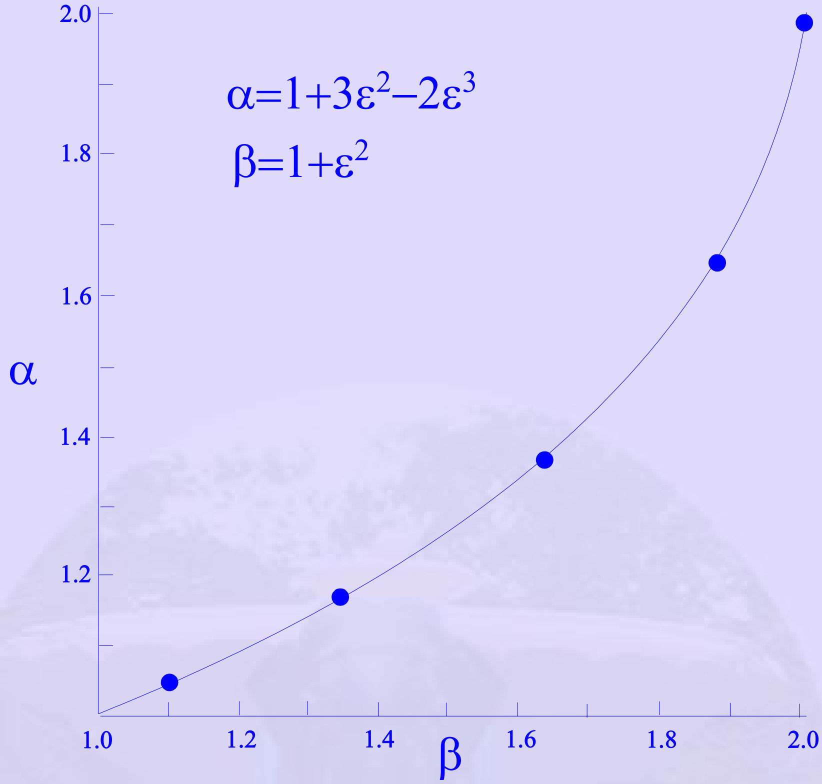
Substituting for " ε " in the expression for " α ",

$$\alpha = 1 + 3\left(\sqrt{(\beta - 1)^2}\right) - 2\left(\sqrt{\beta - 1}\right)^3 = 1 + 3(\beta - 1) - 2(\beta - 1)^{3/2}$$

$=> \alpha = 3\beta - 2(\beta - 1)^{3/2} - 2$

β	α
1.0	1
1.1	1.237
1.2	1.421
1.3	1.571
1.4	1.694
1.5	1.793
1.6	1.8705
1.7	1.929
1.8	1.969
1.9	1.992
2.0	2
2.1	1.993
2.2	1.971
2.3	1.9355
2.4	1.887
2.5	1.826
2.6	1.752
2.7	1.667
2.8	1.57

The plot is shown below



Relationship between α and β

6.2.1 Derivation of relationships

Assuming a wide channel with the two - dimensional velocity distribution given by

$$\frac{v}{V_0} = \sin\frac{\pi y}{2y_o} \text{ and}$$

$$\frac{v}{V_0} = \left[\frac{y}{y_o}\right]^n$$

determine " α " and " β " (as a function of exponent n in second case). Hence show

that (a) For laminar case $\frac{\alpha - 1}{\beta - 1} = 2.76$ and

(b) for turbulent case $\frac{\alpha - 1}{\beta - 1} = \frac{(n+3)(2n+1)}{(3n+1)}$.

Solution:

Case (a) : $\frac{v}{V_0} = \sin\frac{\pi y}{2y_o}$

where v is the velocity at a depth of " y " from boundary, y_o is the total depth of flow in wide channel. Let B the width of wide channel.

$$V = V_0 \sin\frac{\pi y}{2y_o}$$

Mean velocity = $\overline{V} = \frac{1}{A}\int v\,dA$

$$\overline{V} = \frac{1}{By_o}\int_0^y V_o \sin\frac{\pi y}{2y_o}\ B\,dy = \frac{V_0}{y_0}\int_0^{y_o} \sin\frac{\pi y}{2y_o}\,dy$$

$$= \frac{V_0}{y_0}\left\{\frac{-2y_o}{\pi}\cos\frac{\pi y}{2y_o}\right\}_0^{y_o}$$

$$= \frac{-2V_o}{\pi}\left\{\cos\frac{\pi y}{2y_o}\right\}_0^{y_o} = \frac{-2V_o}{\pi}\left\{\cos\frac{\pi}{2} - \cos(0)\right\}$$

$$= \frac{-2V_o}{\pi}\{0-1\}$$

$$\boxed{\overline{V} = \frac{2V_o}{\pi}}$$

6.2.2 Kinetic Energy Correction Factor

$$\alpha = \frac{1}{\overline{V}^3 A}\int v^3 \, dA = \frac{1}{\left\{\frac{2V_o}{\pi}\right\}^3 By_o}\int_0^{y_o} V_0^3 \sin^3\frac{\pi y}{2 y_o} B\, dy$$

$$= \frac{\pi^3}{8 y_o}\int_0^{y_o} \sin^3 \frac{\pi y}{2 y_o}\, dy$$

$$\sin^3\frac{\pi y}{2 y_o} = \sin\frac{\pi y}{2 y_o}\sin^2\frac{\pi y}{2 y_o}$$

$$\cos\frac{\pi y}{y_o} = 1 - 2\sin^2\frac{\pi y}{2 y_o} \qquad \cos 2A = 1 - 2\sin^2 A$$

$$\therefore \sin^2\frac{\pi y}{2 y_o} = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi y}{y_o}$$

$$\therefore \sin^3\frac{\pi y}{2 y_o} = \sin\frac{\pi y}{2 y_o}\sin^2\frac{\pi y}{2 y_o} = \sin\frac{\pi y}{2 y_o}\left\{\frac{1}{2} - \frac{1}{2}\cos\frac{\pi y}{y_o}\right\}$$

$$= \frac{1}{2}\sin\frac{\pi y}{2 y_o} - \frac{1}{2}\sin\frac{\pi y}{2 y_o}\cos\frac{\pi y}{2 y_o}$$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\therefore \sin\frac{\pi y}{2 y_o}\cos\frac{\pi y}{2 y_o} = \frac{\sin\left\{\frac{\pi y}{2 y_o} + \frac{\pi y}{y_o}\right\} + \sin\left\{\frac{\pi y}{2 y_o} - \frac{\pi y}{y_o}\right\}}{2}$$

$$= \frac{1}{2}\sin\frac{3\pi y}{2 y_o} + \frac{1}{2}\sin\left\{\frac{-\pi y}{2 y_o}\right\}$$

$$= \frac{1}{2}\sin\frac{3\pi y}{2 y_o} - \frac{1}{2}\sin\frac{\pi y}{2 y_o} \qquad \sin(-A) = -\sin A$$

$$\therefore \sin^3\frac{\pi y}{2 y_o} = \frac{1}{2}\sin\frac{\pi y}{2 y_o} - \frac{1}{2}\sin\frac{\pi y}{2 y_o}\cos\frac{\pi y}{2 y_o}$$

$$= \frac{1}{2}\sin\frac{\pi y}{2 y_o} - \frac{1}{4}\sin\frac{3\pi y}{2 y_o} + \frac{1}{4}\sin\frac{\pi y}{2 y_o}$$

$$\therefore \alpha = \frac{\pi^3}{8 y_o}\int_0^{y_0}\sin^3\frac{\pi y}{2 y_o}\, dy$$

:

$$= \frac{\pi^3}{8y_o}\left\{\frac{1}{2}\int_0^{y_0}\sin^3\frac{\pi y}{2y_o}dy - \frac{1}{4}\int_0^{y_0}\sin^3\frac{3\pi y}{2y_o}dy + \frac{1}{4}\int_0^{y_0}\sin^3\frac{\pi y}{2y_o}dy\right\}$$

$$= \frac{\pi^3}{8y_o}\left\{\frac{1}{2}\left[\frac{-2y_o}{\pi}\cos\frac{\pi y}{2y_o}\right]_0^{y_o} - \frac{1}{4}\left[\frac{-2y_o}{3\pi}\cos^3\frac{\pi y}{2y_o}\right]_0^{y_o} + \frac{1}{4}\left[\frac{-2y_o}{\pi}\cos\frac{\pi y}{2y_o}\right]_0^{y_o}\right\}$$

$$= \frac{\pi^3}{8y_o}\left\{-\frac{y_o}{\pi}(0-1) + \frac{y_o}{6\pi}(0-1) - \frac{y_o}{2\pi}(0-1)\right\}$$

$$= \frac{\pi^3}{8y_o}\left\{\frac{y_o}{\pi} - \frac{y_o}{6\pi} + \frac{y_o}{2\pi}\right\} = \frac{\pi^3}{8y_o}\left\{\frac{6y_o - y_o - 3y_o}{6\pi}\right\}$$

$$= \frac{\pi^3}{8y_o}\;\frac{8y_o}{6\pi} = \frac{\pi^2}{6}$$

$$\therefore \boxed{\alpha = \frac{\pi^2}{6}}$$

6.2.3 Momentum correction factor

$$\beta = \frac{1}{\overline{V}^2 A}\int V^2 dA = \frac{1}{\left\{\frac{2V_0}{\pi}\right\}^2 B\, y_o}\int_0^{y_o} V_0^2 \sin^2\frac{\pi\, y}{2\, y_o}\, B\, dy$$

$$= \frac{\pi^2}{4y_o}\int_0^{y_o}\sin^2\frac{\pi\, y}{2\, y_o}dy$$

$$\cos 2A = 1\text{-}2\sin^2 A$$

$$\therefore \sin^2\frac{\pi\, y}{2\, y_o} = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi y}{y_0}$$

$$\therefore \beta = \frac{\pi^2}{4\, y_o}\int_0^{y_o}\sin^2\frac{\pi\, y}{2\, y_o}dy$$

$$= \frac{\pi^2}{4y_o}\left\{\frac{1}{2}\int_0^{y_o}dy - \frac{1}{2}\int_0^{y_o}\cos\frac{\pi y}{y_0}dy\right\}$$

$$= \frac{\pi^2}{4y_o}\left\{\frac{1}{2}[y]_0^{y_0} - \frac{1}{2}\left[\frac{y_o}{\pi}\sin\frac{\pi y}{y_0}\right]_0^{y_0}\right\}$$

$$= \frac{\pi^2}{4y_o}\left\{\frac{1}{2}(y_o - 0) - \frac{y_o}{2\pi}(0-0)\right\}$$

$$= \frac{\pi^2}{4y_o}\ \frac{y_o}{y}$$

$$\therefore \boxed{\beta = \frac{\pi^2}{8}}$$

$$\therefore \frac{\alpha - 1}{\beta - 1} = \frac{\frac{\pi^2}{6} - 1}{\frac{\pi^2}{8} - 1} = \frac{\pi^2 - 6}{6}\ \frac{8}{\pi^2 - 8} = 2.76$$

case (b): $\frac{v}{V_o} = \left\{\frac{y}{y_o}\right\}^n$

where v is the velocity at a depth " y " from boundary, y_o is the total depth of wide channel.

Let B the width of wide channel

$$v = V_o\left\{\frac{y}{y_o}\right\}^n$$

Mean velocity = $\overline{V} = \frac{1}{A}\int v.dA$

$$\therefore\ \overline{V} = \frac{1}{B\,y_o}\int_0^{y_o} V_o\frac{y^n}{y_0^n}B\,dy = \frac{V_o}{(y_0)^{n+1}}\int_0^{y_o} y^n dy$$

$$= \frac{V_o}{(y_o)^{n+1}}\left[\frac{(y)^{n+1}}{n+1}\right]_0^{y_o} = \frac{V_o}{(y_o)^{n+1}}\left[\frac{(y_o)^{n+1}}{n+1} - 0\right]$$

$$=> \boxed{\overline{V} = \frac{V_o}{n+1}}$$

Kinetic energy correction factor :

$$\alpha = \frac{1}{\overline{V}^3 A}\int v^3 dA = \frac{1}{\left\{\frac{V_o}{n+1}\right\}^3 B\,y_0}\int_0^{y_0} V_0^3\frac{y^{3n}}{y_o^{3n}}B\,dy$$

$$= \frac{(n+1)^3}{y_o y_0^{3n}}\int_0^{y_n} y^{3n}\,dy$$

$$=> \frac{(n+1)^3}{y_0^{3n+1}}\left[\frac{y^{3n+1}}{3n+1}\right]_0^{y_o} = \frac{(n+1)^3}{y_0^{3n+1}}\left[\frac{y^{3n+1}}{3n+1} - 0\right]$$

$$=> \boxed{\alpha = \frac{(n+1)^3}{3n+1}}$$

Momentum correction factor

$$\beta = \frac{1}{\overline{V}^2 A}\int V^2 dA = \frac{1}{\left\{\frac{V_o}{n+1}\right\}^2 B\, y_o}\int_0^{y_o} V_0^2 \frac{y^{2n}}{y_0^{2n}} B\, dy$$

$$= \frac{(n+1)^2}{y_o y_o^{2n}}\int_0^{y} y^n\, dy$$

$$= \frac{(n+1)^2}{y_o^{2n+1}}\left[\frac{y^{2n+1}}{2n+1}\right]_0^{y_0}$$

$$= \frac{(n+1)^2}{y_o^{2n+1}}\left[\frac{y_0^{2n+1}}{2n+1} - 0\right]$$

$$=> \boxed{\beta = \frac{(n+1)^2}{2n+1}}$$

$$\therefore \frac{\alpha-1}{\beta-1} = \frac{\frac{(n+1)^3}{3n+1}-1}{\frac{(n+1)^2}{2n+1}-1} = \frac{\frac{n^3+3n^2+3n+1-3n-1}{(3n+1)}}{\frac{n^2+2n+1-2n-1}{(2n+1)}}$$

$$\boxed{\frac{\alpha-1}{\beta-1} = \frac{(n+3)(2n+1)}{(3n+1)}}$$

If $n = \frac{1}{7}$

$$\alpha = \frac{(n+1)^3}{3n+1} = \frac{\left\{\frac{1}{7}+1\right\}^3}{3*\frac{1}{7}+1}$$

$$=> \alpha = \frac{1.4927}{1.4285} = 1.0449$$

$$\beta = \frac{(n+1)^2}{2n+1} = \frac{\left\{\frac{1}{7}+1\right\}^2}{2*\frac{1}{7}+1}$$

$$=> \beta = \frac{1.3061}{1.2857} = 1.0158$$

Example:

Obtain α and β for the velocity distribution given below

$$u = 0.4 + 0.6\,\frac{y}{h},\ \ h=1.0,$$

Solution:

$$u = \int \frac{1}{h}(u dy) = \frac{1}{1}\int_0^1 \left(0.4 + 0.6\frac{y}{h}\right)$$

$$= \left[(0.4\,y) + \left(0.6\,\frac{y^2}{2}\right)\right]_0^1$$

$$= 0.7 \text{ m/s}$$

$$\alpha = \frac{1}{u^3 h}\int_0^h \left(u^3 dy\right) = \frac{1}{0.7^3 * 1}\int_0^1 (0.4+0.6y)^3 dy$$

$$\alpha = \frac{1}{0.343}\int_0^1 \left(0.064+0.216y^3+0.432y^2+0.288y\right) dy$$

$$= \frac{1}{0.343}\left[0.064y+0.216y^4+0.432y^3+0.288y^2\right]_0^1$$

$$\alpha = 1.18$$

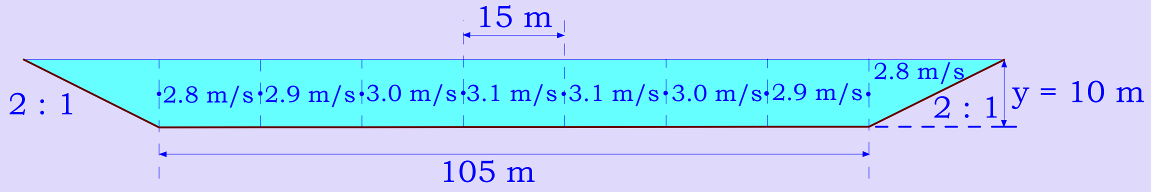
Problems:

1. The velocity distribution (in m/s) in an open channel 2m deep can be represented by the equation,

$$\mathbf{v(y) = 0.6 + 1.4\left(\frac{y}{y_0}\right)^{1/2}}$$

Calculate the energy correction factor. Here in y is the height above bed and y_o = 2m.

2. In a channel of trapezoidal cross section the velocities were measured at mid depth at various sub areas. Compute the average values of α and β for a given cross sections.



3. For an assumed velocity distribution $V = 5.75 V_* \log\left(\frac{30y}{K}\right)$ Prove that $\alpha = 1 + 3\in^2 - 2\in^3$ *and* $\beta = 1 + \in^2$ in which $\in = \frac{V_{max}}{\overline{V}} - 1$, V_{max} is the maximum velocity, $\overline{V}$ is the mean velocity.