Module 3: Behaviour of Laminae

Note:

1. **In this module text in "Italic" indicates advanced concepts.**
2. σ_{xy}, τ_{xy} **are used for 'Shear' in books and literature.**

M3.1 Stress and strain concepts in 3-D

M3.1.1 Stress Concepts:

The Definition of Stress:

The concept of stress originated from the study of strength and failure of solids. The **stress field** is the distribution of internal "tractions" that balance a given set of external tractions and body forces.

First, we look at the external traction 'T' that represents the force per unit area acting at a given location on the body's surface. **Traction** 'T' is a bound vector, which means 'T' cannot slide along its line of action or translate to another location and keep the same meaning.

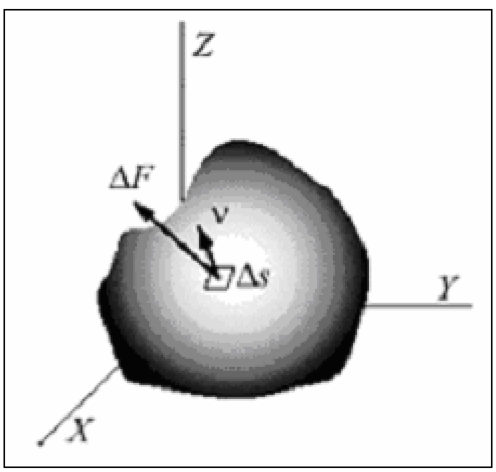


Figure M3.1.1: Action of force (F) on a body

In other words, a traction vector cannot be fully described unless both the force (F) and the surface (S) where the force acts on have been specified, as shown in Figure M3.1.1 Given both ΔF and Δs, the traction 'T' can be defined as,

$$T = \lim_{\Delta s \to 0} \frac{\Delta F}{\Delta s} = \frac{dF}{ds} \quad \textbf{(M3.1.1)}$$

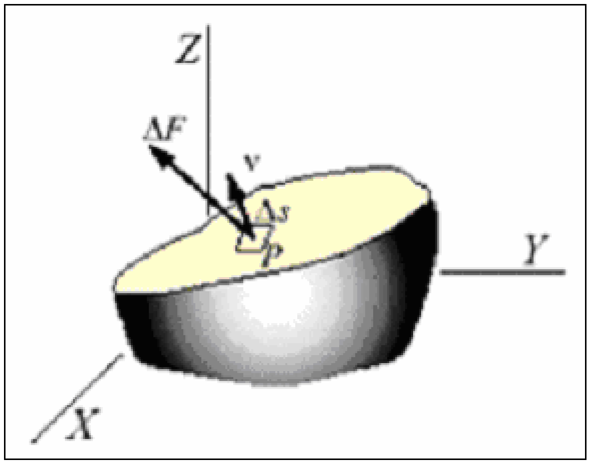


Figure M3.1.2: Stresses at a points

The internal traction within a solid, or stress, can be defined in a similar manner. Suppose an arbitrary slice is made across the solid shown in the Figure M3.1.2, leading to the free body diagram shown at right. Surface tractions would appear on the exposed surface, similar in form to the external tractions applied to the body's exterior surface. The stress at point ‘P’ can be defined using the same equation as was used for ‘T’.

Stress therefore can be interpreted as internal tractions that act on a defined internal datum plane. One cannot measure the stress without first specifying the datum plane.

Components of Stress: The Stress Tensor (or Stress Matrix):

Surface tractions, or stresses acting on an internal datum plane, are typically decomposed into three mutually orthogonal components. One component is normal to the surface and represents direct stress. The other two components are tangential to the surface and represent shear stresses.

What is the distinction between normal and tangential tractions, or equivalently, direct and shear stresses? Direct stresses tend to change the volume of the material (e.g. hydrostatic pressure) and are resisted by the body's bulk modulus (which depends on the Young's modulus and Poisson ratio). Shear stresses tend to deform the material without changing its volume, and are resisted by the body's shear modulus.

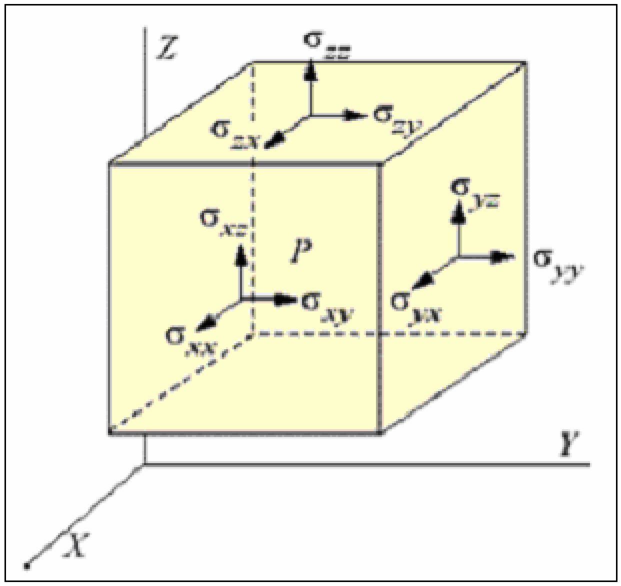


Figure M3.1.3: Elements 3-dimensional stress. All stresses have positive sense.

Defining a set of internal datum planes aligned with a Cartesian coordinate system allows the stress state at an internal point 'P' to be described relative to x-, y-, and z-coordinate directions as shown in Figure M3.1.3.

For example, the stress state at point 'P' can be represented by an infinitesimal cube with three stress components on each of its six sides (one direct and two shear components).

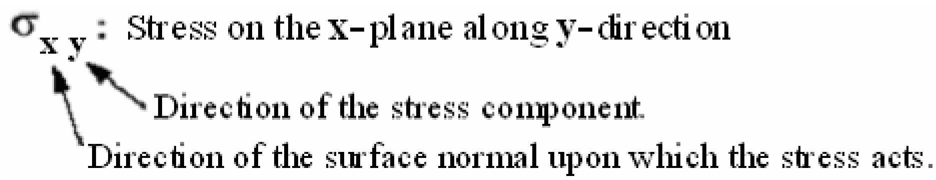
Since each point in the body is under static equilibrium (no net force in the absence of any body forces), only nine stress components from three planes are needed to describe the stress state at a point 'P'.

These nine components can be organized into the matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad \textbf{(M3.1.2)}$$

where shear stresses across the diagonal are identical (i.e. $\sigma_{xy} = \sigma_{yx}, \sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$) as a result of **static equilibrium** (no net moment). This grouping of the nine stress components is known as the stress tensor (or stress matrix).

The subscript notation used for the nine stress components have the following meaning:



Note:

The stress state is a second order tensor since it is a quantity associated with two directions. As a result, stress components have 2-subscripts. A surface traction is a first order tensor (i.e. vector) since it a quantity associated with only one direction. Vector components therefore require only 1 subscript. Mass would be an example of a zero-order tensor (i.e. scalars), which have no relationships with directions (and no subscripts).

Equations of Equilibrium:

Consider the static equilibrium of a solid subjected to the body force vector field b. Applying Newton's first law of motion results in the following set of differential equations which govern the stress distribution within the solid,

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \\ \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases} \qquad \textbf{(M3.1.3)}$$

In the case of two dimensional stress, the above equations reduce to,

$$\begin{cases} \dfrac{\partial \sigma_{x}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{y}}{\partial y} + b_y = 0 \end{cases} \qquad \textbf{(M3.1.4)}$$

M3.1.1 Strain Concepts:

Global 1D Strain:

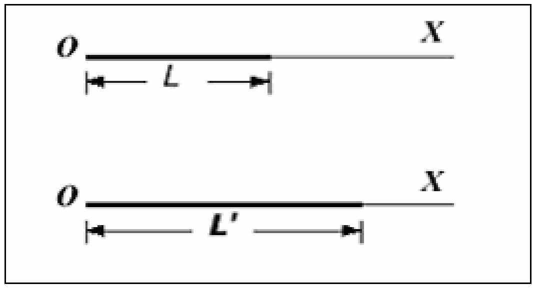


Figure M3.1.4: Global 1-dimensional strain.

Consider a rod with initial length L which is stretched to a length L', as shown in Figure M3.1.4. The strain measure ε, a dimensionless ratio, is defined as the ratio of elongation with respect to the original length,

$$\varepsilon = \frac{L' - L}{L} \qquad \textbf{(M3.1.5)}$$

Infinitesimal 1D Strain:

The above strain measure is defined in a global sense. The strain at each point may vary dramatically if the bar's elastic modulus or cross-sectional area changes. To track down the strain at each point, further refinement in the definition is needed.

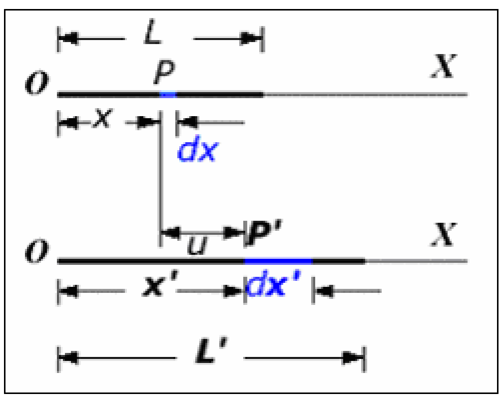


Figure M3.1.5: Infinitesimal 1-dimensional strain.

Consider an arbitrary point in the bar P, which has a position vector x, and its infinitesimal neighbor 'dx'. Point 'P' shifts to P', which has a position vector x', after the stretch; as shown in Figure M3.1.5. In the meantime, the small "step" dx is stretched to dx'.

The strain at point p can be defined the same as in the global strain measure,

$$\varepsilon = \frac{dx' - dx}{dx} \qquad \textbf{(M3.1.6)}$$

Since the displacement $u = x - x'$, the strain can hence be rewritten as,

$$\varepsilon = \frac{dx' - dx}{dx} = \frac{du}{dx} \qquad \textbf{(M3.1.7)}$$

General Definition of 3D Strain:

As in the one dimensional strain derivation, suppose that point P in a body shifts to point 'P' after deformation, as shown in Figure M3.1.5.

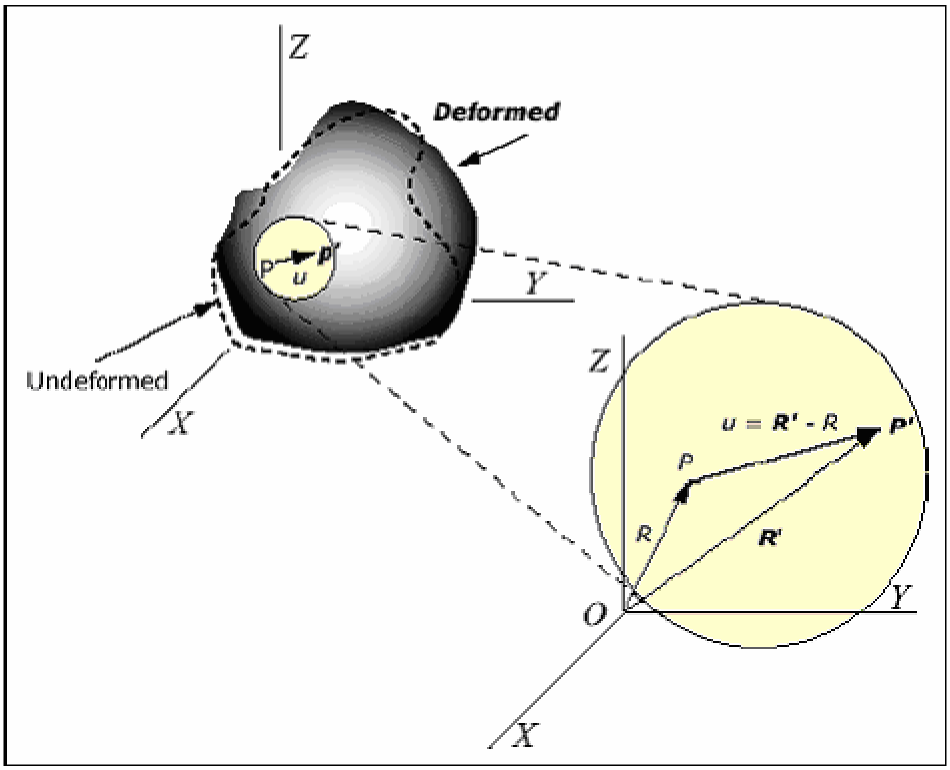


Figure M3.1.4: General definition of 3D-strain

The infinitesimal strain-displacement relationships can be summarized as,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \textbf{(M3.1.8)}$$

where ' u ' is the displacement vector, 'x' is coordinate, and the two indices 'i' and 'j' can range over the three coordinates $\{1,2,3\}$ in three dimensional space.

Expanding the above equation for each coordinate direction gives,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \varepsilon_{zy}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \varepsilon_{xz} \qquad \textbf{(M3.1.9)}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \varepsilon_{yx}$$

where u, v, and w are the displacements in the x, y, and z directions respectively (i.e. they are the components of u).

3D Strain Matrix:

There are a total of six - strain measures. These six - measures can be organized into a matrix (similar in form to the 3D stress matrix), shown here,

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \textbf{(M3.1.10)}$$

Engineering Shear Strain:

Focus on the strain ε_{xy} for a moment. The expression inside the parentheses can be rewritten as,

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \textbf{(M3.1.11(a))}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$. Called the **engineering shear strain**, γ_{xy} is a total measure of shear strain in the x-y plane. In contrast, the shear strain ε_{xy} (in Figure M3.1.7 (a)) is the average of the shear strain on the 'x' face along the y-direction, and on the 'y' face along the x-direction (in Figure M3.1.7 (b)).

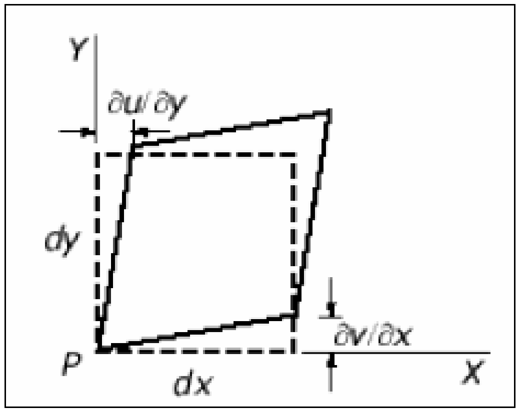


Figure M3.1.7 (a): Shear strain tensor is the average of strains, i.e.

$$\varepsilon_{xy} = (\partial v / \partial x + \partial u / \partial y)/2 = \varepsilon_{yx} \qquad \textbf{(M3.1.11(b))}$$

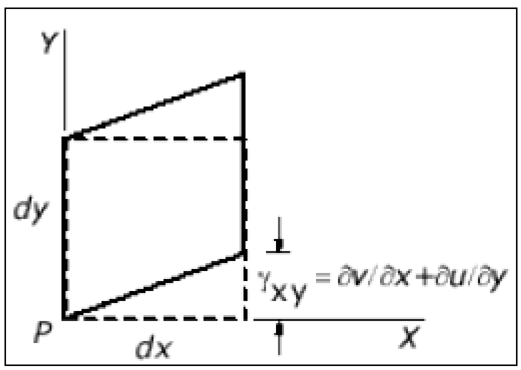


Figure M3.1.7 (b): Engineer shear strain is the total of strains, i.e.

$$\gamma_{xy} = (\partial v / \partial x + \partial u / \partial y) \qquad \textbf{(M3.1.11(c))}$$

Engineering shear strain is commonly used in engineering reference books. However, please beware of the difference between shear strain and engineering shear strain, so as to avoid errors in mathematical manipulations.

Compatibility Conditions:

In the strain-displacement relationships, there are six strain measures but only three independent displacements. That is, there are 6-unknowns for only 3-independent variables. As a result there exist 3-constraint, or compatibility, equations.

These compatibility conditions for infinitesimal strain refered to rectangular Cartesian coordinates are,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \qquad \textbf{(M3.1.12)}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \qquad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

In two dimensional problems (e.g. plane strain), all 'z' terms are set to zero. The compatibility equations reduce to,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \textbf{(M3.1.13)}$$

Note that some references use engineering shear strain ($\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$) when referencing compatibility equations.

M3.1.2.One-dimensional Hooke's Law:

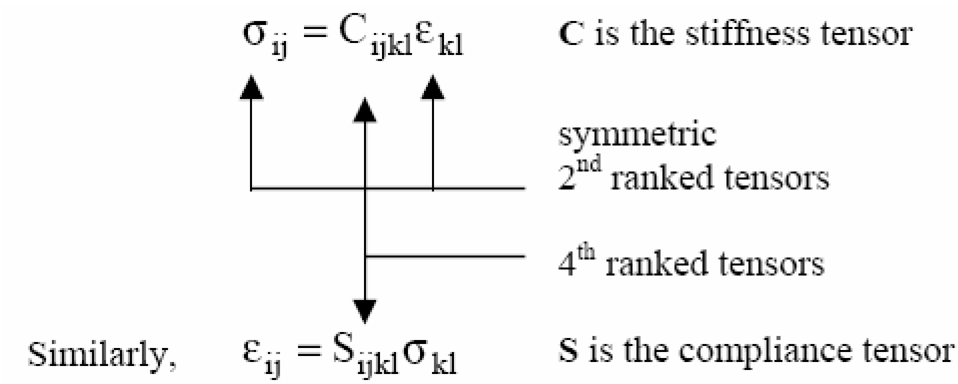
Robert Hooke (who in 1676) stated that, "The power of any springy body is in the same proportion with the extension" and it commonly called/announced as "the birth of elasticity".

Hooke's statement expressed mathematically is,

$$F = k.u \qquad \textbf{(M3.1.14)}$$

where 'F' is the applied force (and not the power, as Hooke mistakenly suggested), 'u' is the deformation of the elastic body subjected to the force F, and k is the spring constant (i.e. the ratio of previous two parameters).

Generalized Hooke's Law (Anisotropic Form):



S is the inverse of C. These equations encompass all anisotropic crystalline behavior.

Note: In this module text in "Italic" indicates advanced concepts.

Number of elements:

C (or S) have i, j, k and l = 1, 2, 3. Therefore, C has $3\times3\times3\times3=3^4 = 81$ *terms. Under general transformations,* $C'_{ijkl} = a_{im}a_{jn}a_{ko}a_{lp}C_{mnop}$. *There are* 3^4 *terms from the direction cosines and* 3^4 *terms from the 'C' tensor, or 6561 terms needed!*

Symmetry Considerations:

Recall that σ_{ij} *and* ε_{ij} *are symmetric matrices, i.e.* $\sigma_{ij} = \sigma_{ji}$. *This has consequences for the stiffness and compliance tensors. With (1)* $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$, *we can swap dummy labels (k, l) and then, because of symmetry of strain tensor, we find (2)* $\sigma_{ij} = C_{ijlk}\varepsilon_{lk} = C_{ijlk}\varepsilon_{kl}$. *Thus,* $C_{ijlk} = C_{ijlk}$. *With similar swap on stress tensor (i, j) subscripts and use of symmetry, we get* $C_{jikl} = C_{ijkl}$. *Thus, instead of 81, there are only 36-independent coefficients. The same symmetries hold for Compliance, 'S'.*

Lastly, the strain energy per unit volume for a body with dilation (volume change) and distortion (deformation) is simply, $U = \frac{1}{2}\sigma_{ij}\varepsilon_{ji}$. *This work must be independent of which strains were performed first. By a theorem of homogeneous functions, this means that 2^nd^ derivative of U with respect to strain should be independent of which order the derivative is taken. Consider the*

Differential Work: $\delta U = \sigma_{ij}\delta\varepsilon_{ji} = C_{ijkl}\varepsilon_{kl}\delta\varepsilon_{ji}$. *So,* $\frac{\delta U}{\delta\varepsilon_{ji}} = C_{ijkl}\varepsilon_{kl}$ *and, therefore,* $\frac{\delta^2 U}{\delta\varepsilon_{ji}\delta\varepsilon_{kl}} = C_{ijkl} = \frac{\delta^2 U}{\delta\varepsilon_{kl}\delta\varepsilon_{ij}} = C_{klij}$. *The symmetry* $C_{klij} = C_{ijkl}$ *means there is only 21-independent coefficients.*

Summary of Symmetries:

- $C_{ijlk} = C_{ijkl}$ *from symmetry of strains*
- $C_{jikl} = C_{ijkl}$ *from symmetry of stresses*
- $C_{klij} = C_{ijkl}$ *from energy considerations*

Non-isotropic, Linear Elastic Behavior

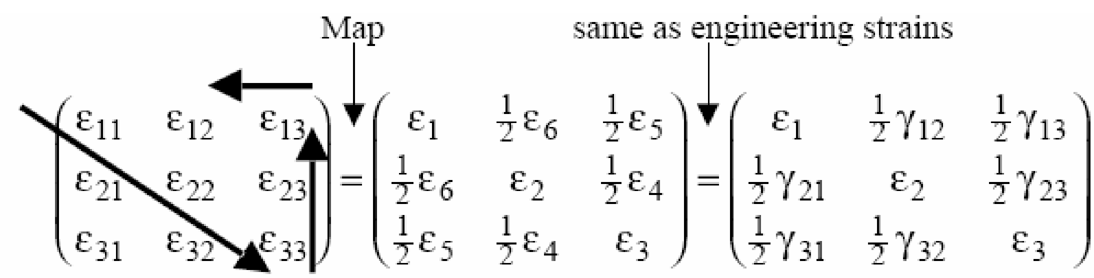
Contracted Notation:

Unlike a matrix (which has rows and columns, 2-D representation), general tensors cannot be written on the blackboard. So in order to manipulate the elastic relations like a matrix, a mapping was created to write the stress and strain tensors as "vectors" and the stiffness (or compliance) as a "matrix". The mapping is simply:

Tensor subscripts:	**11**	**22**	**33**	**23,32**	**13,31**	**12,21**
Matrix subscripts:	**1**	**2**	**3**	**4**	**5**	**6**

E.G., $C_{1111}=C_{11}$, $C_{1122}=C_{12}$, $C_{1133}=C_{13}$, $C_{1123}=C_{14}$, $C_{1113}=C_{15}$, $C_{1112}=C_{16}$.

Memory Aide:



$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \frac{1}{2}\varepsilon_6 & \frac{1}{2}\varepsilon_5 \\ \frac{1}{2}\varepsilon_6 & \varepsilon_2 & \frac{1}{2}\varepsilon_4 \\ \frac{1}{2}\varepsilon_5 & \frac{1}{2}\varepsilon_4 & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_3 \end{pmatrix}$$

The "vector" of strains is assigned along arrowed path. Also the engineering strains have a factor of two in definition (as required), which cancels the 2's in equation (M3.1.15) below.

A real example,

$$\sigma_{11} = C_{11kl}\varepsilon_{kl} = C_{1111}\varepsilon_{11} + C_{1112}(2\varepsilon_{12}) + C_{1113}(2\varepsilon_{13}) + C_{1122}\varepsilon_{22} + C_{1123}(2\varepsilon_{23}) + C_{1133}\varepsilon_{33} \quad \textbf{(M3.1.15)}$$

$$\sigma_1 \underset{\text{MAP}}{\overset{}{=}} C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{14}\varepsilon_4 + C_{15}\varepsilon_5 + C_{16}\varepsilon_6 \quad \textbf{(M3.1.16)}$$

Generally,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \quad \textbf{(M3.1.17)}$$

This "matrix" of stiffness constants has 36-elements, but only 21-independent because it is a symmetric "matrix", as found by symmetry of the C tensor (see previous page).

Non-isotropic, Linear Elastic Behavior

For the contraction of strain relation, i.e., $\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \Rightarrow S_{ij}\sigma_j$*, a little more care must be taken. The rules may be simple stated:*

1. $S_{ijkl} = S_{mm}$ *when* ***both*** *'m' and 'n' are 1, 2, or 3.*
2. $2S_{ijkl} = S_{mm}$ *when* ***either*** *'m' and 'n' are 4, 5, or 6.*
3. $4S_{ijkl} = S_{mm}$ *when* ***both*** *'m' and 'n' are 4, 5, or 6*

Best explained by two examples:

Example 1:

In tensor notation, ε_{11} *is given in terms of stresses as,*

$$\begin{aligned}\varepsilon_{11} = \quad & S_{1111}\sigma_{11} + S_{1112}\sigma_{12} + S_{1113}\sigma_{13} \\ & + S_{1121}\sigma_{21} + S_{1122}\sigma_{22} + S_{1123}\sigma_{23} \\ & + S_{1131}\sigma_{31} + S_{1132}\sigma_{32} + S_{1133}\sigma_{33} \quad = S_{11kl}\sigma_{kl}\end{aligned} \quad \textbf{\textit{(M3.1.18)}}$$

In contracted notation

$$\begin{aligned}\varepsilon_1 = \quad & S_{11}\sigma_1 + \tfrac{1}{2}S_{16}\sigma_6 + \tfrac{1}{2}S_{15}\sigma_5 \\ & + \tfrac{1}{2}S_{16}\sigma_6 + \quad S_{12}\sigma_2 + \tfrac{1}{2}S_{14}\sigma_4 \\ & + \tfrac{1}{2}S_{15}\sigma_5 + \tfrac{1}{2}S_{14}\sigma_4 + \quad S_{13}\sigma_3 \\ & = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6 \quad = S_{1j}\sigma_j\end{aligned} \quad \textbf{\textit{(M3.1.19)}}$$

Example 2:

$$\begin{aligned}\varepsilon_{23} = \quad & S_{2311}\sigma_{11} + S_{2312}\sigma_{12} + S_{2313}\sigma_{13} \\ & + S_{2321}\sigma_{21} + S_{2322}\sigma_{22} + S_{2323}\sigma_{23} \\ & + S_{2331}\sigma_{31} + S_{2332}\sigma_{32} + S_{2333}\sigma_{33} \quad = S_{23kl}\sigma_{kl}\end{aligned} \quad \textbf{\textit{(M3.1.20)}}$$

In contracted notation, and remembering that 12 ε ε = 23 , and rules (2) and (3) above

$$\begin{aligned}\tfrac{1}{2}\varepsilon_4 = \; & \tfrac{1}{2}S_{41}\sigma_1 + \tfrac{1}{4}S_{46}\sigma_6 + \tfrac{1}{4}S_{45}\sigma_5 \\ & + \tfrac{1}{4}S_{46}\sigma_6 + \tfrac{1}{2}S_{42}\sigma_2 + \tfrac{1}{4}S_{44}\sigma_4 \\ & + \tfrac{1}{4}S_{45}\sigma_5 + \tfrac{1}{4}S_{44}\sigma_4 + \tfrac{1}{2}S_{43}\sigma_3\end{aligned} \quad \textbf{\textit{(M3.1.21)}}$$

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6-components of stress are linearly related to the 6-components of strain.

The stress-strain relationship written in matrix form, where the 6-components of stress and strain are organized into column vectors, is,

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}, \quad \varepsilon = \mathbf{S}\cdot\sigma \qquad \textit{(M3.1.22)}$$

or,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}, \quad \sigma = \mathbf{C}\cdot\varepsilon \qquad \textit{(M3.1.23)}$$

where C is the stiffness matrix, 'S' is the compliance matrix, and $S = C^{-1}$.

In general, stress-strain relationships such as these are known as constitutive relations.

In general, there are 36-stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21-stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the stiffness matrix is traditionally represented by the symbol 'C', while 'S' is reserved for the compliance matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

M3.2 Introduction to Anisotropic Elasticity

Stress-strain relationships for Isotropic, Orthotropic and Transverse materials

M3.2.1 Isotropic Definition:

Most metallic alloys and thermoset polymers are considered isotropic, where by definition the material properties are independent of direction. Such materials have only 2 independent variables (i.e. elastic constants) in their stiffness and compliance matrices, as opposed to the 21 elastic constants in the general anisotropic case.

The two elastic constants are usually expressed as the **Young's modulus E** and the **Poisson's ratio** η (or 'n'). However, the alternative elastic constants **bulk modulus (**K**)** and/or **shear modulus (**G**)** can also be used. For isotropic materials, G and K can be found from E and n by a set of equations, and vice-versa.

Hooke's Law in Compliance Form:

Hooke's law for isotropic materials in compliance matrix form is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} \quad \textbf{(M3.2.1)}$$

Some literatures may have a factor 2 multiplying the shear moduli in the compliance matrix resulting from the difference between shear strain and engineering shear strain, where, $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$ etc.

Hooke's Law in Stiffness Form:

The stiffness matrix is equal to the inverse of the compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} \quad \textbf{(M3.2.2)}$$

Some literatures may have a factor 1/2 multiplying the shear moduli in the stiffness matrix resulting from the difference between shear strain and engineering shear strain, where, $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$ etc.

M3.2.2 Orthotropic Definition:

Some engineering materials, including certain piezoelectric materials (e.g. Rochelle salt) and 2-ply fiber-reinforced composites, are orthotropic.

By definition, an orthotropic material has at least 2-orthogonal planes of symmetry, where material properties are independent of direction within each plane. Such materials require 9-independent variables (i.e. elastic constants) in their constitutive matrices.

In contrast, a material without any planes of symmetry is fully anisotropic and requires 21-elastic constants, whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry) is isotropic, and requires only 2-elastic constants.

Hooke's Law in Compliance Form:

By convention, the 9-elastic constants in orthotropic constitutive equations are comprised of 3-Young's moduli E_x, E_y, E_z, the 3-Poisson's ratios $\nu_{yz}, \nu_{zx}, \nu_{xy}$, and the 3-shear moduli G_{yz}, G_{zx}, G_{xy}.

The compliance matrix takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} \quad \textbf{(M3.2.3)}$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}, \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}, \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses $\sigma_x', \sigma_y', \sigma_z$ and the shear strains $\varepsilon_{yz}', \varepsilon_{zx}', \varepsilon_{xy}$.

The factor 1/2 multiplying the shear moduli in the compliance matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form:

The stiffness matrix for orthotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{yz}\nu_{zy}}{E_y E_z \Delta} & \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta} & \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xy}+\nu_{xz}\nu_{zy}}{E_z E_x \Delta} & \frac{1-\nu_{zx}\nu_{xz}}{E_z E_x \Delta} & \frac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta} & \frac{\nu_{yz}+\nu_{xz}\nu_{yx}}{E_x E_y \Delta} & \frac{1-\nu_{xy}\nu_{yx}}{E_x E_y \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_{xy} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} \quad \textbf{(M3.2.4)}$$

where,

$$\Delta = \frac{1-\nu_{xy}\nu_{yx} - \nu_{yz}\nu_{zy} - \nu_{zx}\nu_{xz} - 2\nu_{xy}\nu_{yx}\nu_{zx}}{E_x E_y E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$
\begin{cases}
\dfrac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta}=\dfrac{\nu_{xy}+\nu_{xz}\nu_{zx}}{E_z E_x \Delta} \\
\dfrac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta}=\dfrac{\nu_{zy}+\nu_{xz}\nu_{yz}}{E_x E_y \Delta} \\
\dfrac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta}=\dfrac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta}
\end{cases}
\qquad \textbf{(M3.2.5)}
$$

The factor of 2-multiplying the shear moduli in the stiffness matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

M3.2.3 Transverse Isotropic Definition:

A special class of orthotropic materials is those that have the same properties in one plane (e.g. the x-y plane) and different properties in the direction normal to this plane (e.g. the z-axis). Such materials are called transverse isotropic, and they are described by 5-independent elastic constants, instead of 9 for fully orthotropic.

Examples of transversely isotropic materials include some piezoelectric materials (e.g. PZT-4, barium titanate) and fiber-reinforced composites where all fibers are in parallel.

Hooke's Law in Compliance Form:

By convention, the 5-elastic constants in transverse isotropic constitutive equations are the Young's modulus and Poisson's ratio in the x-y symmetry plane, E_p and ν_p, the Young's modulus and Poisson's ratio in the z-direction, E_{pz} and ν_{pz}, and the shear modulus in the z-direction G_{zp}.

The compliance matrix takes the form,

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_p} & -\frac{\nu_p}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_p}{E_p} & \frac{1}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{pz}}{E_p} & -\frac{\nu_{pz}}{E_p} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu_p}{E_p} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} \qquad \textbf{(M3.2.6)}
$$

where $\dfrac{\nu_{pz}}{E_p} = \dfrac{\nu_{zp}}{E_z}$.

The factor 1/2- multiplying the shear moduli in the compliance matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form:

The stiffness matrix for transverse isotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}\frac{1-\nu_{pz}\nu_{zp}}{E_pE_z\Delta} & \frac{\nu_p+\nu_{zp}\nu_{pz}}{E_pE_z\Delta} & \frac{\nu_{zp}+\nu_p\nu_{zp}}{E_pE_z\Delta} & 0 & 0 & 0\\ \frac{\nu_p+\nu_{pz}\nu_{zp}}{E_zE_p\Delta} & \frac{1-\nu_{zp}\nu_{pz}}{E_zE_p\Delta} & \frac{\nu_{zp}+\nu_{zp}\nu_p}{E_zE_p\Delta} & 0 & 0 & 0\\ \frac{\nu_{pz}+\nu_p\nu_{pz}}{E_p{}^2\Delta} & \frac{\nu_{pz}(1+\nu_p)}{E_p{}^2\Delta} & \frac{1-\nu_p{}^2}{E_p{}^2\Delta} & 0 & 0 & 0\\ 0 & 0 & 0 & 2G_{zp} & 0 & 0\\ 0 & 0 & 0 & 0 & 2G_{zp} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{E_p}{1+\nu_p}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix} \quad \textbf{(M3.2.7)}$$

where,

$$\Delta=\frac{(1+\nu_p)(1-\nu_p-2\nu_{pz}\nu_{zp})}{E_p{}^2E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases}\frac{\nu_p+\nu_{zp}\nu_{pz}}{E_pE_z\Delta}=\frac{\nu_p+\nu_{pz}\nu_{zp}}{E_zE_p\Delta}\\ \frac{\nu_{zp}+\nu_{zp}\nu_p}{E_zE_p\Delta}=\frac{\nu_{pz}(1+\nu_p)}{E_p{}^2\Delta}\\ \frac{\nu_{zp}+\nu_p\nu_{zp}}{E_pE_z\Delta}=\frac{\nu_{pz}+\nu_p\nu_{pz}}{E_p{}^2\Delta}\end{cases} \quad \textbf{(M3.2.8)}$$

These three equations are the counterparts of $\frac{\nu_{pz}}{E_p}=\frac{\nu_{zp}}{E_z}$ in the compliance matrix.

The factor of 2- multiplying the shear moduli in the stiffness matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy}=\varepsilon_{xy}+\varepsilon_{yx}=2\varepsilon_{xy}$, etc.

Note: In this module text in "Italic" indicates advanced concepts.

Independent Material Constants:

Hooke was probably the first person that suggested a mathematical expression of the stress-strain relation for a given material.

The most general stress-strain relationship (a.k.a. generalized Hooke's law) within the theory of linear elasticity is that of the materials without any plane of symmetry, i.e., general anisotropic materials or triclinic materials. If there is a plane of symmetry, the material is termed monoclinic. If the number of symmetric planes increases to two, the third orthogonal plane of material symmetry will automatically yield and form a set of principal axes. In this case, the material is known as orthotropic. If there exists, a plane in which the mechanical properties are equal in all directions, the material is called transversely isotropic. If there is an infinite number of planes of material symmetry, i.e., the mechanical properties in all directions are the same at a given point, the material is known as isotropic.

Please distinguish 'isotropic' from 'homogeneous.' A material is isotropic when its mechanical properties remain the same in all directions at a given point while they may change from point to point; a material is homogeneous when its mechanical properties may be different along different directions at given point, but this variation is consistent from point to point. For example, consider three common items on a dining table: stainless steel forks, bamboo chopsticks, and Swiss cheese. Stainless steel is isotropic and homogeneous. Bamboo chopsticks are homogeneous but not isotropic (they are transversely isotropic, strong along the fiber direction, relatively weak but equal in other directions). Swiss cheese is isotropic but not homogeneous
(The air bubbles formed during production left inhomogeneous spots).

Both stress and strain fields are second order tensors. Each component consists of information in two directions: the normal direction of the plane in question and the direction of traction or deformation. There are 9-components in each field in a three dimensional space. Since they are symmetric, engineers usually rewrite them from a 3×3 matrix to a vector with 6-components and arrange the stress-strain relations into a 6×6 matrix to form the generalized Hooke's law. For the 36 components in the stiffness or compliance matrix, not every component is independent to each other and some of them might be zero. This information is summarized in the following table.

	Independent Constants	***Nonzero On-axis***	***Nonzero Off-axis***	***Nonzero General***
Triclinic (General Anisotropic)	*21*	*36*	*36*	*36*
Monoclinic	*13*	*20*	*36*	*36*
Orthotropic	*9*	*12*	*20*	*36*
Transversely Isotropic	*5*	*12*	*20*	*36*
Isotropic	*2*	*12*	*12*	*12*

A more detailed discussion of stress, strain, and the stress-strain relations of materials can be found in the Mechanics of Materials section.

Learning Unit-3: M3.3

M3.3 Tensorial concept and indicial notation and tensorial representations in Elasticity, Voigot notations

M3.3 Tensorial Concept and Indicial Notations

M3.3.1 Tensorial Concept/'General Characteristics of Tensors'

In the study of **particle mechanics** and/or the **mechanics of rigid bodies (Solid Mechanics)**, vector notation provides a convenient means for describing many physical quantities and laws. In studying the mechanics of (solid) deformable media, physical quantities of a more complex nature assume importance; e.g., stress and strain (recall: one must now specify not only the magnitude of the quantity, but also the orientation of the face upon which this quantity acts). Mathematically such physical quantities are represented by matrices.

In the analysis of general problems in **continuum mechanics**, the physical quantities encountered can be somewhat more complex than vectors and matrices. Like vectors and matrices these physical quantities are independent of any particular coordinate system that may be used to describe them. At the same time, these physical quantities are very often specified most conveniently by referring to an appropriate system of coordinates. Tensors (which are a generalization of vectors and matrices) offer a suitable way of representing these quantities mathematically.

As an abstract mathematical entity, tensors have an existence independent of any coordinate system (or frame of reference), yet are most conveniently described by specifying their components in an appropriate system of coordinates. Specifying the components of a tensor in one coordinate system determines the components in any other system. Indeed, the law of transformation of tensor components is often used as a means for defining the tensor.

Since such transformations are linear and homogeneous, tensor equations valid in one coordinate system are valid in any other coordinate system. This constitutes the invariance of tensor equations under coordinate transformations.

M3.3.1.2 Concept of Tensor Rank

Tensors may be classified by rank or order according to the particular form of transformation law they obey. This classification is also reflected in the number of components a given tensor possesses in an N-dimensional space. Thus, a tensor of order 'n' has N^n components. For example, in a three-dimensional Euclidean space, the number of components of a tensor is '3n'. It follows therefore, that in three-dimensional space:

- A tensor of order zero has one component and is called a scalar (physical quantities possessing magnitude only are represented by scalars).
- A tensor of order one has three components and is called a vector (quantities possessing both magnitude and direction are represented by vectors). Geometrically, vectors are represented by directed line segments which obey the Parallelogram Law of addition.
- A tensor of order two has nine components and is typically represented by a matrix.

Notation

The following symbols are used in the sequel:

- Scalars are represented by lowercase Greek letters; e.g., α ;
- Vectors are represented by lowercase Latin letters; e.g., **a** , $\underset{\sim}{a}$ or {a } ; and,
- Matrices and tensors are represented by uppercase Latin letters; e.g., **A**, $\underset{\sim}{A}$ or [A].

M3.3.1.2 Concept of Cartesian Tensor

When only transformations from one homogeneous coordinate system (e.g., a Cartesian coordinate system) to another are considered, the tensors involved are referred to as Cartesian tensors. The Cartesian coordinate system can be rectangular (x, y, z) or curvilinear such as cylindrical (r,θ, z) or spherical (r,θ,φ).

Tensors are quantities that are governed by specific transformation laws when the coordinate system is transformed. If the reference coordinate frame is Cartesian then the associated tensors are known as Cartesian Tensors. Tensors can be classified according to their order.

Zero order tensor → Scalar

First order tensor → Vector

Second order tensor → Dyadic (ex. stress, strain)

In general, a Cartesian tensor of n^{th} order has $3n$ components. It is not possible to show a simple geometrical representation of a tensor when the order is higher than one.

M3.3.2 Indicial Notation

A tensor of any order, and/or its components, may be represented clearly and concisely by the use of indicial notation (this convention was believed to have been introduced by Einstein). In this notation, letter indices, either subscripts or superscripts, are appended to the generic or kernel letter representing the tensor quantity of interest; e.g., A_{ij}, B_{ijk} , δ_{ij} , a_{kl} , etc.

Tensors of order higher than one are often encountered in the solution of applied mechanics problems. For example, three-dimensional problems involve all nine components of stress and strain tensors. In addition for general anisotropy, the stress-strain relationship is defined by a fourth order tensor. Formulation of 3-D problems in mechanics involves situations where algebraic manipulations become quite tedious. Indicial notation provides a compact means to express and manipulate mathematical expressions involving tensors.

A vector $\mathbf{A}$, which has three Cartesian components, (A_1, A_2, A_3) is denoted in indicial notation by A_i where the index 'i' is understood to take the values 1, 2, 3 for 3-D problems and 1, 2 for 2-D problems.

A second order tensor or dyadic T, which has nine components, is denoted by T_{ij} with $i,j=1, 2, 3$.

A third order tensor D, which has 27 components, is identified by T_{ijk} with $i, j, k = 1, 2, 3$.

Some benefits of using indicial notation include:

(1) Economy in writing; and,
(2) Compatibility with computer languages (e.g. do loops, etc.).

M3.3.2.1 Rules for Indicial Notation

Some rules for using indicial notation follow.

M3.3.2.1.1 Index rule

In a given term, a letter index may occur no more than twice.

M3.3.2.1.2 Range convention

When an index occurs unrepeated in a term, that index is understood to take on the values $1, 2, \ldots\ldots\ldots, N$, where '**N**' is a specified integer which, depending on the space considered, determines the range of the index.

M3.3.2.1.3 Summation convention

When an index appears twice in a term, that index is understood to take on all the values of its range, and the resulting terms are summed. For example, $A_{kk} = A_{11} + A_{22} + \ldots\ldots + A_{NN}$.

M3.3.2.1.4 Free indices

By virtue of the range convention, unrepeated indices are free to take the values over the range; i.e., $1, 2, \ldots\ldots\ldots, N$. These indices are thus termed "free".

NOTES:

- Any equation must have the same free indices in each term.
- The tensorial rank of a given term is equal to the number of free indices.

- $N^{\text{Number of Free indices}}$ = Number of Components represented by the symbol.

M3.3.2.1.5 Dummy indices

In the summation convention, repeated indices are often referred to as dummy indices, since their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur.

Example 1:

In the following equations the repeated indices constitute "dummy" indices.

$$A_{kk} = A_{mm}$$
$$B_{ik}C_{kl} = B_{in}C_{nl} \qquad \textbf{(M3.3.1)}$$

Example 2:

In the equation,

$$E_{ij} = \varepsilon_{im}\varepsilon_{mj} \qquad \textbf{(M3.3.2)}$$

'i' and 'j' represent free indices and 'm' is a dummy index. Assuming N = 3 and using the range convention, it follows that,

$$E_{ij} = \varepsilon_{i1}\varepsilon_{1j} + \varepsilon_{i2}\varepsilon_{2j} + \varepsilon_{i3}\varepsilon_{3j} \qquad \textbf{(M3.3.3)}$$

Care must be taken to avoid breaking grammatical rules in the indicial "language". For example, the expression,

$$\mathbf{a} \bullet \mathbf{b} = \left(a_k\hat{e}_k\right) \bullet \left(b_k\hat{e}_k\right) \qquad \textbf{(M3.3.4)}$$

is erroneous since the summation on the dummy indices is ambiguous. To avoid such ambiguity, a dummy index can only be paired with one other dummy index in an expression. A good rule to follow is use separate dummy indices for each implied summation in an expression.

M3.3.2.1.6 Contraction of indices

Contraction refers to the process of summing over a pair of repeated indices. This reduces the order of a tensor by two.

Contracting the indices of A_{ij} (a second order tensor) leads to A_{kk} (a zeroeth (0^{th}) order tensor;

i.e., no free indices);

Contracting the indices of B_{ijk} (a third order tensor) leads to B_{imm} (a first order tensor; i.e., one free index);

Contracting the indices of C_{ijkl} (a fourth order tensor) leads to C_{ijmm} (a second order tensor; i.e., two free indices);

M3.3.2.1.7 Comma Subscript Convention

A subscript comma followed by a subscript index 'i' indicates partial differentiation with respect to each coordinate x_i . For example,

$$\phi_{,m} \equiv \frac{\partial \phi}{\partial x_m} \qquad \textbf{(M3.3.5)}$$

$$A_{i,j} \equiv \frac{\partial A_i}{\partial x_j} \qquad \textbf{(M3.3.6)}$$

$$B_{i,jk} \equiv \frac{\partial^2 B_i}{\partial x_j \, \partial x_k} \qquad \textbf{(M3.3.7)}$$

If 'i' remains a free index, differentiation of a tensor with respect to 'i' produces a tensor of order one higher (e.g., see equation M3.3.6). If i is a dummy index, differentiation of a tensor with respect to i produces a tensor of order one lower. For example

$$V_{m,m} = \frac{\partial V_m}{\partial x_m} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \cdots + \frac{\partial V_N}{\partial x_N} \qquad \textbf{(M3.3.8)}$$

Voigot Notations:

Tensorial and Contracted Notation:

Tensorial				Contracted
σ_{11}				σ_1
σ_{22}				σ_2
σ_{33}				σ_3
σ_{23}	=	τ_{23}	=	σ_4 or τ_4
σ_{31}	=	τ_{31}	=	σ_5 or τ_5
σ_{12}	=	τ_{12}	=	σ_6 or τ_6

(M3.3.9)

Learning Unit-3: M3.4

M3.4 Plane stress concept and assumption

The Plane-Stress Assumption

Historically, one of the most important assumptions regarding the study of the mechanics of fiber-reinforced materials is that the properties of the fibers and the properties of the matrix can be smeared into an equivalent homogeneous material with orthotropic material properties. Without this assumption, we would still have to deal with the response of the individual fibers embedded in matrix material, as was done in the study of micromechanics in the last chapter. If this assumption had not been made in the development of the mechanics of fiber-reinforced materials, very little progress would have been made in understanding their response. An equally important assumption in the development of the mechanics of fiber-reinforced materials is the plane-stress assumption, which is based on the manner in which fiber-reinforced composite materials are used in many structures. Specifically, fiber-reinforced materials are utilized in beams, plates, cylinders, and other structural shapes which have at least one characteristic geometric dimension an order of magnitude less than the other two dimensions. In these applications, three of the six components of stress are generally much smaller than the other three. With a plate, for example, the stresses in the plane of the plate are much larger than the stresses perpendicular to that plane. In all calculations, then, the stress components perpendicular to the plane of the structure can be set to zero, greatly simplifying the solution of many problems. In the context of fiber-reinforced plates, for example, the stress components σ_3, τ_{23} and τ_{13} are set to zero with the assumption that the 1-2 plane of the principal material coordinate system is in the plane of the plate. Stress components σ_1, σ_2 and τ_{12} are considered to be much larger in magnitude than components σ_3, τ_{23} and τ_{13}. In fact, σ_1 should be the largest of all the stress components if the fibers are being utilized effectively. We use the term plane stress because σ_1, σ_2 and τ_{12} lie in a plane, and stresses σ_3, τ_{23} and τ_{13} are perpendicular to this plane and are zero.

The plane-stress assumption can lead to inaccuracies, some serious and some not so serious. The most serious inaccuracy occurs in the analysis of a laminate near its edge. Laminates tend to come apart in the thickness direction, or delaminate, at their edges, much like common plywood. An understanding of this phenomenon, illustrated in Figure M3.4.2, requires that all six components of stress be included in the analysis. It is exactly the stresses that are set to zero in the plane-stress assumption (i.e. σ_3, τ_{23} and τ_{13}) that are responsible for delamination, so an analysis that ignores these stresses cannot possibly be correct for a delamination study. Delaminations can also occur away from a free edge, with the layers separating in blister fashion. These are generally caused by the presence of imperfections between the layers. The out-of-plane stress components σ_3, τ_{23} and τ_{13} are also important in locations where structures or components of structures are joined together; Figures M3.4.3 and M3.4.4 illustrate some examples. Figure M3.4.3 shows a bonded joint consisting of two laminates subjected to tensile load 'P'. For the load to be transferred from one laminate to the other, significant out-of-plane stresses, particularly shear, must develop in the laminates around the interface, as well as at the interface itself. As another example, in many situations stiffeners are used to increase the load capacity of plates, as in Figure M3.4.4. For the plate-stiffener combination to be effective, the plate must transfer some of the pressure load to the stiffener. Thickness direction stresses must develop in the plate and stiffener flange if load is to be transferred through the interface. In general, all three components of out-of-plane stress σ_3, τ_{23} and τ_{13} develop in this situation. Away from the stiffener the plate may be in a state of plane stress, so not only is there a region of the plate characterized by a fully three-dimensional stress state, there is also a transition region. In this transition region the conditions go from truly plane stress to a fully three-dimensional stress state, making the analysis of such a problem difficult and challenging.

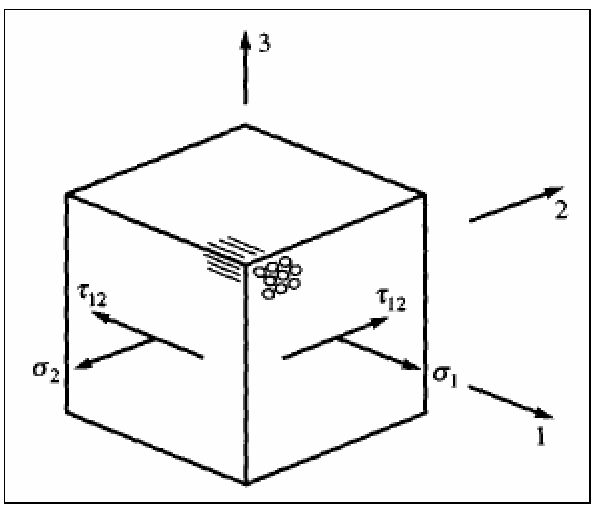


Figure M3.4.1 Stresses acting on a small element of fiber-reinforced material in a state of plane stress

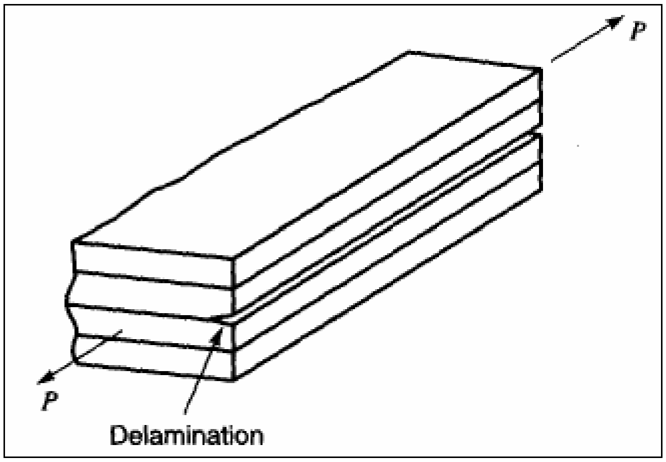


Figure M3.4.2 Example of region of high out-of-plane stresses: Delamination at a free edge

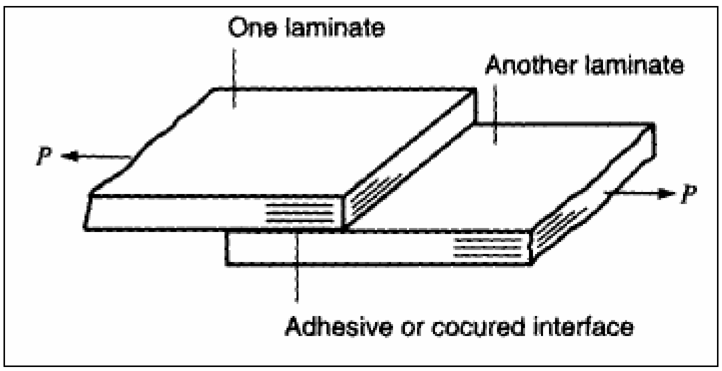


Figure M3.4.3 Another laminate Example of region of high out-of-plane stresses: Bonded joint

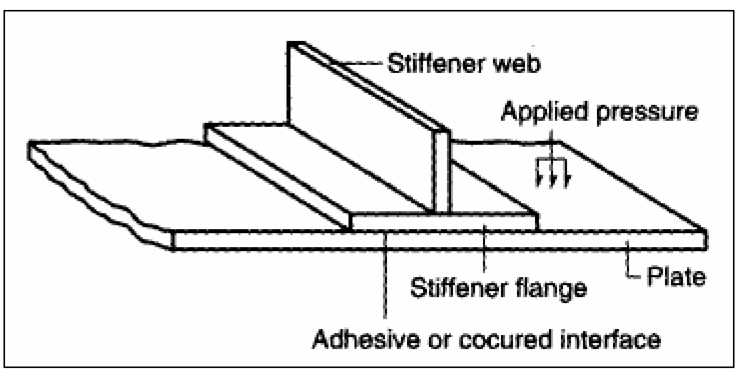


Figure M3.4.4 Example of region of high out-of-plane stresses: Stiffened plate

Figure M3.4.5 illustrates another area where through-the-thickness stresses are important. Often it is necessary, or desirable, to change the thickness of a laminate by gradually terminating some of the layers. Away from the terminated layer region each portion of the laminate could well be in a state of plane stress due to the applied inplane load 'P'. However, the thicker region is in a different state of plane stress than the thinner region. To make the transition between the two stress states, three-dimensional effects occur.

The illustrations in Figures M3.4.2 to Figure M3.4.5 are prime examples of situations encountered in real composite structures. However, the plane-stress assumption is accurate in so many situations that one would be remiss in not taking advantage of its simplifications. The static, dynamic, and thermally induced deflections and the stresses that result from these, vibration frequencies, buckling loads, and many other responses of composite structures can be accurately predicted using the plane-stress assumption. What is important to remember when applying the plane-stress assumption is that it assumes that three stresses are small relative to the other three stresses and they have therefore been set to zero. They do not necessarily have to be exactly zero, and in fact in many cases they are not exactly zero. With the aid of the three-dimensional equilibrium equations of the theory of elasticity, calculations based on the plane-stress assumption can be used to predict the stress components that have been equated to zero. When these results are compared with predictions of the out-of-plane components based on rigorous analyses wherein the out-of-plane components are not assumed to be zero at the outset, we find that in many cases the comparisons are excellent. Thus, a plane-stress, or, using alternative terminology, a two-dimensional analysis, is useful. Two of the major pitfalls associated with using the plane-stress assumption are:

1. The stress components equated to zero are often forgotten and no attempt is made to estimate their magnitude.
2. It is often erroneously assumed that because the stress component σ_3 is zero and therefore ignorable, the associated strain ε_3 is also zero and ignorable.

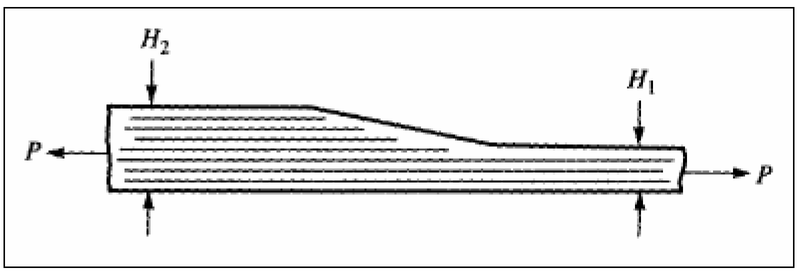


Figure M3.4.5 Example of region of high out-of-plane stresses: Region of terminal layers

Regarding the former point, while certain stress components may indeed be small, the material may be very weak in resisting these stresses. As was stated earlier, a fiber-reinforced material is poor in resisting all stresses except stresses in the fiber direction. Thus, several stress components may be small and so the problem conforms to the plane-stress assumption. However, the out-of-plane stresses may be large enough to cause failure of the material and therefore they should not be completely ignored. Often they are. Regarding the second point, the stresses in the 1-2 plane of the principal material coordinate system can cause a significant strain response in the 3-direction. The assumption that ε_3 is zero just because σ_3 is negligible is wrong and, as we shall see shortly, defies the stress-strain relations that govern material behavior. It is important to keep these two points in mind as we focus our discussion in the following chapters on the plane-stress condition.

M3.4.1 Stress-Strain Relations for Plane Stress

To see why the plane-stress assumption is important, it is only necessary to see how it simplifies the stress-strain relations. Specifically, for the plane-stress assumption σ_3, τ_{23} and τ_{13} are set to zero, then stress-strain relation for plane stress becomes,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix} \quad \textbf{(M3.4.1)}$$

From this relation it is obvious that

$$\gamma_{23} = 0 \qquad \gamma_{13} = 0 \quad \textbf{(M3.4.2)}$$

so with the plane-stress assumption there can be no shear strains whatsoever in the 2-3 and 1-3 planes. That is an important ramification of the assumption. Also,

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 \quad \textbf{(M3.4.3)}$$

This equation indicates explicitly that for a state of plane stress there is an extensional strain in the 3-direction. To assume that strain ε_3 is zero is absolutely wrong. That it is not zero is a direct result of Poisson's ratios ν_{13} and ν_{23} acting through S_{13} and S_{23}, respectively, coupling with the nonzero stress components σ_1 and σ_2. The above equation for ε_3 forms the basis for determining the thickness change of laminates subjected to inplane loads, and for computing through-thickness, or out-of-plane, Poisson's ratios of a laminate.

Despite the fact that ε_3 is not zero, the plane-stress assumption leads to a relation involving only $\varepsilon_1, \varepsilon_2, \gamma_{12}$ and $\sigma_1, \sigma_2, \tau_{12}$. By eliminating the third, fourth, and fifth equations of equation (M3.4.1), we find

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad \textbf{(M3.4.4)}$$

The definitions of the compliances have not changed from the tithe they were first introduced, namely,

$$S_{11} = \frac{1}{E_1} \qquad S_{12} = \frac{-\nu_{12}}{E_1} = \frac{-\nu_{21}}{E_2}$$
$$S_{22} = \frac{1}{E_2} \qquad S_{66} = \frac{1}{G_{12}} \tag{M3.4.5}$$

The 3x 3 matrix of compliances is called the reduced compliance matrix. In matrix notation the lower right hand element of a 3 x 3 matrix is usually given the subscript 33, though in the analysis of composites it has become conventional to retain the subscript convention from the three-dimensional formulation and maintain the subscript of the lower corner element as 66. For an isotropic material, equation (M3.4.5) reduces to

$$S_{11} = S_{22} = \frac{1}{E} \qquad S_{12} = -\frac{\nu}{E} \qquad S_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E} \tag{M3.1.1}$$

If the plane-stress assumption is used to simplify the inverse form of the stress-strain relation, is giving equation (M3.4.7),

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{M3.4.6}$$

With the above, one also concludes that

$$\gamma_{23} = 0 \qquad \gamma_{13} = 0 \tag{M3.4.7}$$

In analogy to equation (M3.4.3), the third equation of equation (M3.4.7) yields,

$$0 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 \tag{M3.4.8}$$

Rearranged, it becomes

$$\varepsilon_3 = -\frac{C_{13}}{C_{33}}\varepsilon_1 - \frac{C_{23}}{C_{33}}\varepsilon_2 \tag{M3.4.9}$$

This relationship also indicates that in this state of plane stress ε_3 exists and equation

(M3.4.10) indicates it can be computed by knowing ε_1 and ε_2.

The three-dimensional form equation (M3.4.7) cannot be reduced directly to obtain a relation involving only $\sigma_1, \sigma_2, \tau_{12}$, in, and $\varepsilon_1, \varepsilon_2, \gamma_{12}$ by simply eliminating equations, as was done with equation (M3.4.1) to obtain equation (M3.4.4). However, equation (M3.4.10) can be used as follows: From equation (M3.4.7), the expressions for σ_1 and σ_2 are

$$\begin{aligned} \sigma_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 \\ \sigma_2 &= C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 \end{aligned} \qquad \textbf{(M3.4.10)}$$

Substituting for ε_3 using equation (M3.4.10) leads to

$$\begin{aligned} \sigma_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\left(-\frac{C_{13}}{C_{33}}\varepsilon_1 - \frac{C_{23}}{C_{33}}\varepsilon_2\right) \\ \sigma_2 &= C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\left(-\frac{C_{13}}{C_{33}}\varepsilon_1 - \frac{C_{23}}{C_{33}}\varepsilon_2\right) \end{aligned} \qquad \textbf{(M3.4.11)}$$

or

$$\begin{aligned} \sigma_1 &= \left(C_{11} - \frac{C_{13}^2}{C_{33}}\right)\varepsilon_1 + \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_2 \\ \sigma_2 &= \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_1 + \left(C_{22} - \frac{C_{23}^2}{C_{33}}\right)\varepsilon_2 \end{aligned} \qquad \textbf{(M3.4.12)}$$

Including the shear stress-shear strain relation, the relation between stresses and strains for the state of plane stress is written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad \textbf{(M3.4.13)}$$

The Q_{ij} are called the reduced stiffnesses and from equations (M3.4.13) and equation (M3.4.7)

$$
\begin{aligned}
Q_{11} &= C_{11} - \frac{C_{13}^2}{C_{33}} \\
Q_{12} &= C_{12} - \frac{C_{13}C_{23}}{C_{33}} \\
Q_{22} &= C_{22} - \frac{C_{23}^2}{C_{33}} \\
Q_{66} &= C_{66}
\end{aligned}
\qquad \textbf{(M3.4.14)}
$$

The term reduced is used in relations given by equations (M3.4.4) and equation (M3.4.14) because they are the result of reducing the problem from a fully three-dimensional to a two-dimensional, or plane-stress, problem. However, the numerical values of the stiffnesses $Q_{11}, Q_{12}, \text{and } Q_{13}$ are actually less than the numerical values of their respective counterparts for a fully three-dimensional problem, namely, $C_{11}, C_{12}, \text{and } C_{33}$ and so the stiffnesses are reduced in that sense also.

It is very important to note that there is not really a numerically reduced compliance matrix. The elements in the plane-stress compliance matrix, equation (M3.4.5), are simply a subset of the elements from the three-dimensional compliance matrix, equation (M3.4.1), and their numerical values are the identical. On the other hand, the elements of the reduced stiffness matrix, equation (M3.4.15), involve a combination of elements from the three-dimensional stiffness matrix. It is absolutely wrong to write,

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}
$$

and claim this represents the reduced stiffness matrix. It simply is not so. By inverting equation (M3.4.4) and comparing it to equation (M3.4.14), it is clear that

$$
\begin{aligned}
Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} \qquad & Q_{12} &= -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} \\
Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} \qquad & Q_{66} &= \frac{1}{S_{66}}
\end{aligned}
\qquad \textbf{(M3.4.15)}
$$

This provides a relationship between elements of the reduced compliance matrix and elements of the reduced stiffness matrix. A much more convenient form, and one that should be used in lieu of equation (M3.4.16), can be obtained by simply writing the compliance components in equation (M3.4.16) in terms of the appropriate engineering constants, namely,

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$
$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \qquad \textbf{(M3.4.16)}$$
$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$
$$Q_{66} = G_{12}$$

This form will be used exclusively from now on. For an isotropic material the reduced stiffnesses become

$$Q_{11} = Q_{22} = \frac{E}{1 - \nu^2} \qquad Q_{12} = \frac{\nu E}{1 - \nu^2} \qquad Q_{66} = G = \frac{E}{2(1 + \nu)} \quad \textbf{(M3.4.17)}$$

M3.4.2 Important Interpretation of Stress-Strain Relations Revisited

When discussing general stress states, we strongly emphasized that only one of the quantities in each of the six stress-strain pairs $\sigma_1 - \varepsilon_1, \sigma_2 - \varepsilon_2, \sigma_3 - \varepsilon_3$ $, \tau_{23} - \gamma_{23}, \tau_{12} - \gamma_{13}$ and $\tau_{12} - \gamma_{12}$ could be specified. With the condition of plane stress, this restriction also holds. For the state of plane stress we assume that σ_3, τ_{23}, and τ_{13} are zero. We can say nothing a priori regarding $\varepsilon_3, \gamma_{23}$, and γ_{13} - However, by using the stress-strain relations; we found, equation (M3.4.2), that is γ_{23}, and γ_{13} are indeed zero. This is a consequence of the **plane-stress condition**, not a stipulation. The strain σ_3 is given by equation (M3.4.3), another consequence of the plane-stress condition. Of the three remaining stress-strain pairs, $\sigma_1 - \varepsilon_1, \sigma_2 - \varepsilon_2$, and $\tau_{12} - \gamma_{12}$ only one quantity in each of these pairs can be specified. The other quantity must be determined, as usual, by using the stress-strain relations, either equation (M3.4.4) or equation (M3.4.14), and the details of the specific problem being solved.