

9. Digital Filters

In many applications of signal processing we want to change the relative amplitudes and frequency contents of a signal. This process is generally referred to as filtering. Since the Fourier transform of the output is product of input Fourier transform and frequency response of the system, we have to use appropriate frequency response.

1. Ideal frequency selective filters:

An ideal frequency reflective filter passes complex exponential signal. for a given set of frequencies and completely rejects the others. Figure (9.1) shows frequency response for ideal low pass filter (LPF), ideal high pass filter (HPF), ideal bandpass filter (BPF) and ideal backstop filter (BSF).

FIGURE 9.1

The ideal filters have a frequency response that is real and non-negative, in other words, has a zero phase characteristics. A linear phase characteristics introduces a time shift and this causes no distortion in the shape of the signal in the passband.

Since the Fourier transfer of a stable impulse response is continuous function of ω , can not get a stable ideal filter.

2. Filter specification:

Since the frequency response of the realizable filter should be a continuous function, the magnitude response of a lowpass filter is specified with some acceptable tolerance. Moreover, a transition band is specified between the passband and stop band to permit the magnitude to drop off smoothly. Figure (9.2) illustrates this

FIGURE 9.2

In the passband magnitude the frequency response is within $\pm\delta_p$ of unity

$$(1 - \delta_p) \leq |H(e^{j\omega})| \leq (1 + \delta_p), \quad |\omega| \leq \omega_p$$

In the stopband

$$|H(e^{j\omega})| \leq \delta_s, \quad |\omega| < \omega_s \leq \pi$$

The frequencies ω_p and ω_s are respectively, called the passband edge frequency and the stopband edge frequency. The limits on tolerances δ_p and δ_s are called the peak ripple value. Often the specifications of digital filter are given in terms of the loss function $G(w) = -20\log_{10}|H(e^{j\omega})|$, in dB. The loss specification of digital filter are

$$\alpha_p = -20\log_{10}(1 - \delta_p)dB$$

$$\alpha_s = -20\log_{10}\delta_s dB$$

Some times the maximum value in the passband is assumed to be unity and the maximum passband deviation, denoted as $\frac{1}{\sqrt{1+E^2}}$ is given the minimum value

of the magnitude in passband. The maximum stopband magnitude is denoted by $1/A$. The quantity α_{max} is given by

$$\alpha_{max} = 20 \log_{10}(\sqrt{1 + E^2}) dB$$

These are illustrated in Fig(9.3)

FIGURE 9.3

If the phase response is not specified, one prefers to use IIR digital filter. In case of an IIR filter design, the most common practice is to convert the digital filter specifications to analog low pass prototype filter specifications, to determine the analog low pass transfer function $H_a(s)$ meeting these specifications, and then to transform it into desired digital filter transfer function $H(z)$. This methods is used for the following reasons:

- (a) Analog filter approximation techniques are highly advanced.
- (b) They usually yield closed form solutions.
- (c) Extensive tables are available for analog-design.
- (d) Many applications require the digital solutions of analog filters.

The transformations generally have two properties (1) the imaginary axis of the s-plane maps into unit circle of the z-plane and (2) a stable continuous time filter is transformed to a stable discrete time filter.

3. Filter design by impulse invariance:

In the impulse variance design procedure the impulse response of the impulse response $\{h[n]\}$ of the discrete time system is proportional to equally spaced samples of the continues time filter, i.e.,

$$h[n] = T_d h_a(nT_d) \tag{9.1}$$

where T_d represents a sampling interval, since the specifications of the filter are given in discrete time domain, it turns out that T_d has no role to play in design of the filter. From the sampling theorem we know that the frequency response of the discrete time filter is given by

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} H_a(j\frac{w}{T_d} + j\frac{2\pi k}{T_d})$$

Since any practical continuous time filter is not strictly bandlimited there is some aliasing. However, if the continuous time filter approaches zero at high frequencies, the aliasing may be negligible. Then the frequency response of the discrete time filter is

$$H(e^{jw}) \approx H_a(j\frac{w}{T_d}), \quad |w| \leq \pi$$

We first convert digital filter specifications to continuous time filter specifications. Neglecting aliasing, we get $H_a(j\Omega)$ specification by applying the relation

$$\Omega = \omega/T_d \tag{9.2}$$

where $H_a(j\Omega)$ is transferred to the designed filter $H(z)$, we again use equation (9.2) and the parameter T_d cancels out.

Let us assume that the poles of the continuous time filter are simple, then

$$H_a(s) = \sum_{k=1}^N \frac{A_k}{s - s_k}$$

The corresponding impulse response is

$$h_a(t) = \begin{cases} \sum_{k=1}^N A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\text{Then } h[n] = T_d h_a(nT_d) = \sum_{k=1}^N T_d A_k e^{s_k n T_d} u[n]$$

The system function for this is

$$H(z) = \sum_{k=1}^N \frac{T_d A_k}{(1 - e^{s_k T_d} z^{-1})}$$

We see that a pole at $s = s_k$ in the s-plane is transformed to a pole at $z = e^{s_k T_d}$ in the z-plane. If the continuous time filter is stable, that is $Re\{s_k\} < 0$, then the magnitude of $e^{s_k T_d}$ will be less than 1, so the pole will be inside unit circle. Thus the causal discrete time filter is stable. The mapping of zeros is not so straight forward.

Example: Design a lowpass IIR digital filter $H(z)$ with maximally flat magnitude characteristics. The passband edge frequency ω_p is 0.25π with a passband ripple not exceeding 0.5dB. The minimum stopband attenuation at the stopband edge frequency ω_s of 0.55π is 15dB.

We assume that no aliasing occurs. Taking $T_d = 1$, the analog filter has $\Omega_p = 0.25\pi$, $\Omega_s = 0.55\pi$, the passband ripple is 0.5dB, and minimum stopped attenuation is 15dB. For maximally flat frequency response we choose Butterworth filter characteristics. From passband ripple of 0.5 dB we get

$$20 \log_{10} |H_a(j 0.25\pi)| = -0.5 \text{ dB}$$

$$|H_a(j\Omega_p)|^2 = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^2} = \frac{1}{1 + \epsilon^2}$$

at passband edge.

From this we get $\epsilon^2 = 0.122$

From minimum stopband attenuation of 15 dB we get

$$|H_a(j\Omega_s)|^2 = \frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^2} = \frac{1}{A^2}$$

at stopped edge $A^2 = 31.62$

The inverse discrimination ratio is given by

$$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\epsilon} = 15.84$$

and inverse transition ratio $1/k$ is given by

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.2$$

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.50$$

Since N must be integer we get $N = 4$. By $(\frac{\Omega_p}{\Omega_c})^{2N} = \epsilon^2$ we get $\Omega_c = 1.02$
The normalized Butterworth transfer function of order 4 is given by

$$H_{an}(s) = \frac{1}{(s^2 + .7654s + 1)(s^2 + 1.8478s + 1)}$$

$$= \frac{-0.92s - 0.707}{s^2 + .7654s + 1} + \frac{0.92s + 1.707}{s^2 + 1.848s + 1}$$

This is for normalized frequency of 1 rad/s. Replace s by $\frac{s}{\Omega_c}$ to get $H_a(s)$, from this we get

$$H(z) = \frac{-0.94 + 0.16z^{-1}}{1 - .79z^{-1} + .45z^{-2}} + \frac{.94 - .00167z^{-1}}{1 - .71Z^{-1} + 0.15Z^{-2}}$$

4. Bilinear Transformation

This technique avoids the problem of aliasing by mapping $j\Omega$ axis in the s-plane to one revaluation of the unit circle in the z-plane.

If $H_a(s)$ is the continues time transfer function the discrete time transfer function is detained by replacing s with

$$s = \frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (9.3)$$

Rearranging terms in equation (9.3) we obtain.

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s} \quad (9.4)$$

Substituting $s = \sigma + j\Omega$, we get

$$z = \frac{1 + \sigma \frac{T_d}{2} + j \frac{\Omega T_d}{2}}{1 - \sigma \frac{T_d}{2} - j \Omega \frac{T_d}{2}}$$

If $\sigma < 0$, it is then magnitude of the real part in denominator is more than that of the numerator and so $|z| < 1$. Similarly if $\sigma > 0$, than $|z| > 1$ for all Ω . Thus poles in the left half of the s-plane will get mapped to the poles inside the unit circle in z-plane. If $\sigma = 0$ then

$$z = \frac{1 + j \frac{\Omega T_d}{2}}{1 - j \frac{\Omega T_d}{2}}$$

So, $|z| = 1$, writing $z = e^{j\omega}$ we get

$$e^{j\omega} = \frac{1 + j\Omega \frac{T_d}{2}}{1 - j\Omega \frac{T_d}{2}}$$

rearranging we get

$$j\Omega \frac{T_d}{2} = \frac{e^{j\omega} - 1}{e^{j\omega} + 1} = \frac{e^{j\omega/2}(e^{+j\omega/2} - e^{-j\omega/2})}{e^{j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})}$$

$$= j \frac{\sin \omega/2}{\cos \omega/2}$$

or

$$\Omega = \frac{2}{T_d} \tan \omega/2 \quad (9.5)$$

or

$$\omega = 2 \tan^{-1} \frac{\Omega T_d}{2} \quad (9.6)$$

The compression of frequency axis represented by (9.5) is nonlinear. This is illustrated in figure 9.4.

FIGURE 9.4

Because of the nonlinear compression of the frequency axis, there is considerable phase distortion in the bilinear transformation.

Example: We use the specifications given in the previous example. Using equation (9.5) with $T_d = 2$ we get

$$\Omega_p = \tan \frac{.25\pi}{2} = 0.414$$

$$\Omega = \tan \frac{.55\pi}{2}$$

5. Some frequently used analog filters

In the previous two examples we have used Butterworth filter. The Butterworth filter of order n is described by the magnitude square frequency response of

$$|H_n(j\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_c})^{2n}}$$

It has the following properties.

1. $|H_n(j\Omega)|^2 = 1$ at $\Omega = 0$
2. $|H_n(j\Omega)|^2 = 1/2$ at $\Omega = \Omega_c$
3. $|H_n(j\Omega)|^2$ is monotonically decreasing function of Ω
4. As n gets larger, $|H_n(j\Omega)|^2$ approaches an ideal low pass filter
5. $|H_n(j\Omega)|^2$ is called maximally flat at origin, since all order derivative exist and they are zero at $\Omega = 0$

The poles of a Butterworth filter lie on circle of radius Ω_c in s -plane.

There are two types of Chebyshev filters, one containing ripples in the passband (type I) and the other containing a ripple in the stopband (type II). A Type I low pass normalizer Chebyshev filter has the magnitude squared frequency response.

$$|H_n(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)}$$

where $T_n(\Omega)$ is n^{th} order Chebyshev polynomial. We have the relationship

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n > 2$$

with $T_0(x) = 1, T_1(x) = x$.

Chebyshev filters have the following properties

1. The magnitude squared frequency response oscillates between 1 and $1/(1+\epsilon^2)$ within the passband, the so called equiripple and has a value of $1/(1+\epsilon^2)$ at $\Omega = 1$, the normalized cut off frequency.
2. The magnitude response is monotonic outside the passband including transition and stopband.
3. The poles of the Chebyshev filter lie on an ellipse in s-plane.

An elliptic filter has ripples both in passband and in stopband. The square magnitude frequency response is given by

$$|H_n(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 R_n^2(\Omega)}$$

where $R_n(\Omega)$ is Chebyshev rational function of Ω determined from specified ripple characteristics.

An n^{th} order Chebyshev filter has sharper cutoff than a Butterworth filter, that is, has a narrower transition bandwidth. Elliptic filter provides the smallest transition width.

6. Design of Digital filter using Digital to Digital transformation

There exists a set of transformation that takes a low pass digital filter and turn into highpass, bandpass, bandstop or another lowpass digital filter. These transformations are given in table 9.1. The transformations all take the form of replacing the z^{-1} in $H(z)$ by $g(z^{-1})$, some function of $z - 1$.

Type From	To	Transformation	Design Formula
Low pass cutoff θ_p	Low pass cutoff ω_p	$z^{-1} \rightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \frac{\sin[(\theta_p-\omega_p)/2]}{\sin[(\theta_p+\omega_p)/2]}$
LPF	HPF	$z^{-1} \rightarrow \frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\frac{\cos[(\theta_p-\omega_p)/2]}{\cos[(\theta_p+\omega_p)/2]}$
LPF	BPF	$z^{-1} \rightarrow \frac{z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+\frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+1}$	$\alpha = \frac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $k = \cot[(\omega_2 - \omega_1)/2] \tan \theta_p/2$
LPF	BSF	$z^{-1} \rightarrow \frac{z^{-2}-\frac{2\alpha}{1+k}z^{-1}+\frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2}-\frac{2\alpha}{1+k}z^{-1}+1}$	$\alpha = \frac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $k = \tan[(\omega_2 - \omega_1)/2] \tan \theta_p/2$

Starting with a set of digital specifications and using the inverse of the design equation given in table 9.1, a set of lowpass digital requirements can be established. A LPF digital prototype filter $H_p(z)$ is then selected to satisfy these requirements and the proper digital to digital transformation is applied to give the desired $H(z)$.

Example: Using the digital to digital transformation, find the system function $H(z)$ for a low-pass digital filter that satisfies the following set the requirements

(a) monotone stop and passband (b)-3dB cutoff frequency of 0.5π (c) attenuation at and past 0.75π is at least 15dB.

Because of monotone requirement, a Butterworth filter is selected. The required n is given by

$$n = \frac{\log_{10}[(10^{-3} - 1)/(10^{1.5} - 1)]}{2\log_{10}\{\tan(.5\pi/2)/\tan .75\pi/2\}} = 1.9412$$

rounded to 2.

$$\omega_p = 2 \tan^{-1}[(10^{0.3} - 1)^{-1/2n} \tan(.5\pi/2)] = 0.5\pi$$

For $\theta_p = 1$, $u_p = .5$ we get from table 9.1. $\alpha = -.293$, From standard tables (or MATLAB) we find standard 2^{nd} order Butterworth filter with cut off $\theta_p = 1$ and then apply the digital transform to get

$$H(z) = \frac{(1 + z^{-1})^2}{3.4142 + .5858z^{-2}}$$

7. FIR filter design

In the previous section, digital filters were designed to give a desired frequency response magnitude without regard to the phase response. In many cases a linear phase characteristics is required through the passband of the filter. It can be shown that causal IIR filter cannot produce a linear phase characteristics and only special forms of causal FIR filters can give linear phase.

If $\{h[n]\}$ represents the impulse response of a discrete time linear system a necessary and sufficient condition for linear phase is that $\{h[n]\}$ have finite duration N , that it be symmetric about its mid point, i.e.

$$h[n] = h[N - 1 - n], \quad n = 0, 1, 2, \dots, (N - 1)$$

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{N-1} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} h[n]e^{-j\omega n} + \sum_{n=N/2}^{N-1} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{N/2-1} h[n]e^{-j\omega n} + \sum_{m=0}^{N/2-1} h[m]e^{-j\omega(N-1-m)} \end{aligned}$$

For N even, we get

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \sum_{n=0}^{N/2-1} 2h[n] \cos(\omega(n - (N-1)/2))$$

For N odd

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \left\{ h\left[\frac{N-1}{2}\right] + \sum_{n=0}^{\frac{N-3}{2}} 2h[n] \cos\left[\omega\left(n - \frac{N-1}{2}\right)\right] \right\}$$

For N even we get a non-integer delay, which will cause the value of the sequence to change, [See continuous time implementation of discrete time system, for interpretation of non-integer delay].

One approach to design FIR filters with linear phase is to use windowing.

The easiest way to obtain an FIR filter is to simply truncate the impulse response of an IIR filter. If $\{h_d[n]\}$ is the impulse response of the designed FIR filter, then an FIR filter with impulse response $\{h[n]\}$ can be obtained as follows.

$$h[n] = \begin{cases} h_d[n], & N_1 \leq n \leq N_2 \\ 0, & \text{otherwise} \end{cases}$$

This can be thought of as being formed by a product of $\{h_d[n]\}$ and a window function $\{w[n]\}$

$$\{h[n]\} = \{h_d[n]\}\{w[n]\}$$

where $\{w[n]\}$ is said to be rectangular window and is given by

$$w[n] = \begin{cases} 1, & N_1 \leq n \leq N_2 \\ 0, & \text{otherwise} \end{cases}$$

Using modulation property of Fourier transfer

$$H(e^{j\omega}) = \frac{1}{2\pi} [H_d(e^{j\omega}) \otimes -W(e^{j\omega})]$$

For example if $H_d(e^{j\omega})$ is ideal low pass filter and $\{w[n]\}$ is rectangular window $H(e^{j\omega})$ is measured version of the ideal low pass frequency response $H_d(e^{j\omega})$.
FIGURE 9.5

In general, the index the main lobe of $W(e^{j\omega})$, the more spreading where as the narrower the main lobe (larger N), the closer $|H(e^{j\omega})|$ comes to $|H_d(e^{j\omega})|$. In general, we are left with a trade-off of making N large-enough so that smearing is minimized, yet small enough to allow reasonable implementation. Much work has been done on adjusting $\{w[n]\}$ to satisfy certain main lobe and side lobe requirements. Some of the commonly used windows are give in below.

(a) Rectangular

$$W_R(n) = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Bartlett (or triangle)

$$w_B[n] = \begin{cases} 2n/(N - 1), & 0 \leq n \leq (N - 1)/2 \\ 2 - 2n/(N - 1), & (N - 1)/2 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Hanning

$$w_{Han}[n] = \begin{cases} \frac{1 - \cos[2\pi n/(N - 1)]}{2}, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

(d) **Harming**

$$w_{Ham}[n] = \begin{cases} 0.54 - 0.46 \cos[2\pi n/(N-1)], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

(e) **Blackman**

$$w_{Bl}[n] = \begin{cases} 0.42 - .5 \cos[2\pi n/(N-1)] + 0.08 \cos[4\pi n/(N-1)], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

(f) **Kaiser**

$$w_K[n] = \begin{cases} \frac{I_0\{w_a[(\frac{N-1}{2})^2 - (n - \frac{N-1}{2})^2]^{1/2}\}}{I_0\{w_a(\frac{N-1}{2})\}}. & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

where $I_0(x)$ is modified zero-order Bessel function of the first kind given by

$$\begin{aligned} I_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} d\theta \\ &= 1 + \sum_{n=1}^{\infty} \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} \right]^2 \end{aligned}$$

The main lobe width and first side lobe attenuation increase as we proceed down the window listed above.

An ideal lowpass filter with linear phase and cut off w_c is characterized by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |w| \leq w_c \\ 0, & w_c < |w| \leq \pi \end{cases}$$

The corresponding impulse response is

$$h_d[n] = \frac{\sin[\omega_c(n - \alpha)]}{\pi(n - \alpha)}$$

Since this is symmetric about $n = \alpha$, if we change $\alpha = (N - 1)/2$ and use one of the windows listed above the will get linear phase FIR filter. Transition width and minimum stopped attenuation are listed in the Table 9.3.

Table 9.3

Window	Transition Width	Minimum stopband attenuation
Rectangular	$4\pi/N$	-21db
Bartlett	$8\pi/N$	-25dB
Hanning	$8\pi/N$	-44dB
Hamming	$8\pi/N$	-53dB
Blackman	$12\pi/N$	-74 dB
Kaiser	variable	variable

We first choose a window that satisfies the minimum attenuation. The transition bandwidth is approximately that allows us to choose the value of N. Actual

frequency response characteristic are then calculated and we see if the requirements are met or not. Accordingly N is adjusted parameters for kaiser window are obtained from design formula available for this MATLAB or similar programmes have all these formulas.

There are many computer aided methods available for designing FIR filters such as Parks-Mc Clellan algorithm for equiripple filters.

8 Realizations of Digital Filters

We have many realizations of digital filter. Some of these are now discussed.

Direct Form Realization - An important class of linear time -invariant systems is characterized by the transfer function.

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \quad (9.30)$$

A system with input $\{x[n]\}$ and output $\{y[n]\}$ could be realized by the following constant coefficient difference equation

$$y[n] = - \sum_{k=1}^N a_k y[n - k] + \sum_{k=0}^M b_k x[n - k] \quad (9.31)$$

A realization of the filter using equation (9.31) is shown in figure (9.6)
FIGURE 9.6

The output $y[n]$ is seen to be weighted sum of input $x[n]$ and past inputs $x[n - 1], \dots, x[n - M]$ and past outputs $y[n - 1], \dots, y[n - N]$. Another realization can be obtained by uniting $H(z)$ as product of two transfer functions $H_1(z)$ and $H_2(z)$, where $H_1(z)$ contains only the denominator or poles and $H_2(z)$ contains only the numerator or zeros as follows

$$H(z) = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{1}{1 + \sum_{l=1}^N a_l z^{-l}}$$

$$H_2(z) = \sum_{k=0}^M b_k z^{-k}$$

FIGURE 9.7

The output of the filter is obtained by calculating the intermediate result $\{w[n]\}$ obtained from operating on the input with filter $H_1(z)$ and then operating on $w[n]$ with filter $H_2(z)$. Thus we obtain

$$W(z) = X(z)H_1(z)$$

or

$$w[n] = x[n] - \sum_{k=1}^N a_k w[n-k]$$

and

$$Y(z) = W(z)H_2(z)$$

or

$$y[n] = \sum_{k=0}^M b_k w[n-k]$$

The realization is shown in figure 9.8

FIGURE 9.8

Upon close examination of Fig 9.8, it can be seen that the two branches of delay elements can be combined as they both refer to delayed versions of $w[n]$ and upon simplification, the direct form II canonical realization is obtained as shown in figure 9.9.

FIGURE 9.9

In this form the number of delay element is $\max(M,N)$. It can be shown that this is the minimum number of delay elements that are required to implement the digital filter. This does not mean that this is the best realization. Immunity to roundoff and quantization are very important considerations.

An important special case that is used as building block occurs when $N = M = 2$. Thus $H(z)$ is ratio of two quadratics in z^{-1} , called biquadratic section, and is given by

$$\begin{aligned} H(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \\ &= \frac{b_0(1 + b_1^1 z^{-1} + b_2^1 z^{-2})}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

The alternative form is found to be useful for amplitude scaling for improving performance filter operation. This form is shown in figure 9.10.

FIGURE 9.10

Cascade Realizations: In the cascade realization $H(z)$ is broken into product of transfer functions $H_1(z), H_2(z), \dots, H_k(z)$ each a rational expression in z^{-1} as follows

$$H(z) = H_1(z)H_2(z)\dots H_k(z)$$

FIGURE 9.11

$H(z)$ could be broken up in many ways; however the most common method is to use biquadratic sections. Thus

$$H_k(z) = \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}, \quad k = 1, 2, \dots, K$$

by letting b_{2k} and a_{2k} equal to zero we get bilinear section. Even among the biquadratic sections we have many choices as how we pair poles and zeros. Also

the order of the sections can be different

Example: Final cascade realization of

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{(z - .25)(z^2 - z + 0.5)}$$

Using only real coefficients $H(z)$ can be decompressed as

$$H(z) = \frac{8(z - .1899)(z^2 - .31z + 1.316)}{(z - .25)(z^2 - z + .5)}$$

Divides both numerator and denominator by z^3 and factoring 8 as 2×4 , one possible rearrangement for $H(z)$ is

$$H(z) = \frac{(2 - 0.3799z^{-1})}{(1 - 0.25z^{-1})} \cdot \frac{(4 - 1.24z^{-1} + 5.264z^{-2})}{(1 - z^{-1} + .5z^{-2})}$$

This can be realized as shown is figure 9.12

FIGURE 9.12

Parallel Realizations - The transfer function $H(z)$ could be written as a sum of transfer functions $H_1(z)$, $H_2(z)$, \dots , $H_k(z)$ as follows:

$$H(z) = H_1(z) + H_2(z) + \dots + H_k(z)$$

One parallel form results when $H_k(z)$ are all selected to be of the following form for ($M < N$).

$$H_k(z) = \frac{b_{0k} + b_{1k}z^{-1}}{1 + a_{1k}z^{-1} + a_{2k}z^{-2}}, \quad k = 1, 2, \dots, K$$

If $M \geq N$, we will have a section $H_0(z)$ of FIR filter, obtained by performing long division. Once denominator polynomial has degree more than the numerator polynomial we perform the partial fraction expansion. The resulting structure is shown in figure 9.13.

FIGURE 9.13

Example: Find the parallel form for the filter given in last example.

$$H(z) = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{(1 - .25z^{-1})(1 - z^{-1} + .5z^{-2})}$$

Using MATLAB program or otherwise we get

$$H(z) = 16 + \frac{8}{1 - .25z^{-1}} + \frac{-16 + 20z^{-1}}{1 - z^{-1} + .5z^{-2}}$$

using direct form realization for individual section we get the structure shown in figure 9.14.

FIGURE 9.14

Apart from these there exist a number of other realizations like lattice form, state variable realization etc.