Chapter 6

Discrete fourier series and Discrete fourier transform

In the last chapter we studied fourier transform representation of aperiodic signal. Now we consider periodic and finite duration sequences.

Discrete fourier series Representation if a periodic signal

Suppose that $\widetilde{x}[n]$ is a periodic signal with period N, that is

$$\widetilde{x}[n+N] = \widetilde{x}[n]$$

As is continues time periodic signal, we would like to represent $\widetilde{x}[n]$ in terms of discrete time complex exponential with period N. These signals are given by

$$e^{\frac{j2\pi}{N}kn}, \;\; k = 0, \pm 1, \pm 2, ... \tag{6.1}$$

All these signals have frequencies is that are multiples of the some fundamental frequency, $\frac{2\pi}{N}$, and thus harmonically related.
These are two important distinction between continuous time and discrete time complex exponential. The first one is that harmonically related continuous time complex exponential $e^{j\Omega_0 kt}$ are all distinct for different values of k, while there are only N different signals in the set.
The reason for this is that discrete time complex exponentials which differ in frequency by integer multiple of 2π are identical. Thus

$$\{e^{j\frac{2\pi}{N}kn}\} = \{e^{j\frac{2\pi}{N}(k+N)n}\}$$

So if two values of k differ by multiple of N, they represent the same signal. Another difference between continuous time and discrete time complex exponential is that $\{e^{j\Omega_0 kt}\}$ for different k have period $\frac{2\pi}{\Omega_0|k|}$, which changes with k. In discrete time exponential, if k and N are relative prime than the period is N and not N/k. Thus if N is a prime number, all the complex exponentials given by (6.1) will have period N.
In a manner analogous to the continuous time, we represent the periodic signal $\widetilde{x}[n]$ as

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k] e^{j\frac{2\pi}{N}kn} \tag{6.2}$$

where

$$\widetilde{X}[k] = \sum_{n-0}^{N-1} \widetilde{x}[n] e^{-j\frac{2\pi}{N}kn} \tag{6.3}$$

In equation (6.2) and (6.3) we can sum over any consecutive N values. The equation (6.2) is synthesis equation and equation (6.3) is analysis equation. Some people use the faction $1/N$ in analysis equation. From (6.3) we can see easily that

$$\widetilde{x}[k] = \widetilde{x}[k+N]$$

Thus discrete Fourier series coefficients are also periodic with the same period N.

Example 1:

$$\{\widetilde{x}[n]\} = \{cos\frac{4\pi}{5}\},$$

$$\widetilde{x}[n] = \frac{1}{2}(e^{j\frac{2\pi}{5}.2} + e^{-j\frac{2\pi}{5}.2})$$

So, $\widetilde{X}[2] = \frac{5}{2}$ and $\widetilde{X}[-2] = \frac{5}{2}$, since the signal is periodic with period 5, coefficients are also periodic with period 5, and $\widetilde{x}[k],\ -1 \leq k \leq 1$.
FIGURE
Now we show that substituting equation (6.3) into (6.2) we indeed get $\widetilde{x}[n]$.

$$\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{m=0}^{N-1} \widetilde{x}[m]e^{-j\frac{2\pi}{N}km}\, e^{j\frac{2\pi}{N}kn}$$

interchanging the order of summation we get

$$= \sum_{m=0}^{N-1} \widetilde{x}[m]\frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)k} \tag{6.4}$$

Now the sum

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)k} = 1 \text{ if } n-m \text{ multiple of} N$$

and for $(n-m)$ not a multiple of N this is a geometric series, so sum is

$$\frac{1}{N}\left(\frac{1-e^{j\frac{2\pi}{N}(m-n)N}}{1-e^{j\frac{2\pi}{N}(m-n)}}\right) = 0$$

As m varies from 0 to $N-1$, we have only one value of m namely $m = n$, for which the inner sum if non-zero. So we set the RHS of (6.4) as $\widetilde{x}[n]$.

Properties of Discrete-Time Fourier Series

Here we use the notation similar to last chapter. Let $\{\widetilde{x}[n]\}$ be periodic with period N and discrete Fourier series coefficients be $\{\tilde{X}[k]\}$ then the write

$$\{\widetilde{X}[n]\} \leftrightarrow \tilde{X}[k]$$

where LHS represents the signal and RHS its DFS coefficients

1. Periodicity DFS coefficients:

As we have noted earlier that DFS Coefficients $\{\widetilde{X}[k]\}$ are periodic with period N.

2. Linearity of DFS:

If

$$\{\widetilde{x}[n]\} \leftrightarrow \{\widetilde{X}[k]\}$$
$$\{\widetilde{y}[n]\} \leftrightarrow \{\widetilde{Y}[k]\}$$

If both the signals are periodic with same period N then

$$A\{\widetilde{x}[n]\} + B\{\widetilde{y}[n]\} \leftrightarrow A\{\widetilde{X}[k]\} + B\{\widetilde{Y}[k]\}$$

3. Shift of a sequence:

$$\{\widetilde{x}[n-m]\} \leftrightarrow \{e^{-j\frac{2\pi}{N}mk}\,\widetilde{X}[k]\} \tag{6.5}$$
$$\{e^{j\frac{2\pi}{N}ln}\,\widetilde{x}[n]\} \leftrightarrow \{\widetilde{X}[k-l]\} \tag{6.6}$$

To prove the first equation we use equation (6.3). The DFS coefficients are given by

$$\sum_{n=0}^{N-1} \widetilde{x}[n-m]e^{-j\frac{2\pi}{N}kn}$$

let $n - m = l$, we get

$$= \sum_{l-m}^{N-1-m} \widetilde{x}[l]e^{-j\frac{2\pi}{N}k(m+l)}$$

since $\widetilde{x}[l]$ is periodic we can use any N consecutive values, then

$$= e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-1} \widetilde{x}[l]e^{-j\frac{2\pi}{N}kl}$$

$$= e^{-j\frac{2\pi}{N}km}\,\widetilde{X}[n]$$

We can prove the relation (6.6) in a similar manner starting from equation (6.3)

4. Duality:

From equation (6.2) and (6.3) we can see that synthesis and analysis equation differ only in sign of the exponential and factor $1/N$. If $\{\widetilde{x}[n]\}$ is periodic with period N, then $\{\widetilde{X}[k]\}$ is also periodic with period N. So we can find the discrete fourier series coefficients of $\widetilde{X}[n]$ sequence.
From equation (6.2) we see that

$$N\widetilde{x}[n] = \sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

Thus

$$N\widetilde{x}[-n] = \sum_{k=0}^{N-1} \widetilde{X}[k]e^{-j\frac{2\pi}{N}kn}$$

Interchanging the role of k and n we get

$$N\widetilde{x}[-k] = \sum_{n=0}^{N-1} \widetilde{X}[n]e^{-j\frac{2\pi}{N}kn}$$

comparing this with (6.3) we see that DFS coefficients of $\{\widetilde{X}[n]\}$ are $\{N\widetilde{x}[-k]\}$, the original periodic sequence is reversed in time and multiplied by N. This is known as duality property. If

$$\{\widetilde{x}[n]\} \leftrightarrow \{\widetilde{X}[k]\} \tag{6.7}$$

then

$$\{\widetilde{X}[n]\} \leftrightarrow \{N\widetilde{x}[-k]\} \tag{6.8}$$

5. Complex conjugation of the periodic sequence:

$$\{\widetilde{x}^*[n]\} \leftrightarrow \{\widetilde{X}^*[-k]\}$$

substituting in equation (6.3) we get

$$\begin{aligned} \sum_{n=0}^{N-1} \widetilde{x}^*[n]e^{-j\frac{2\pi}{N}kn} &= \left[\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}(-k)n}\right]^* \\ &= \widetilde{X}^*[-k] \end{aligned}$$

6. Time reversal:

$$\{\widetilde{x}[-n]\} \leftrightarrow \{\widetilde{X}[-k]\}$$

From equation (6.3) we have the DFS coefficient

$$\sum_{n-0}^{N-1} \widetilde{x}[-n]e^{-j\frac{2\pi}{N}kn}$$

putting $m = -n$ we get

$$= \sum_{m=-(N-1)}^{0} \widetilde{x}[m]e^{j\frac{2\pi}{N}km}$$

Since $\widetilde{x}[m]$ is periodic, we can use any N consecutive values

$$\begin{aligned} &= \sum_{m=0}^{N-1} \widetilde{x}[m]e^{j\frac{2\pi}{N}km} \\ &= \widetilde{X}[-k] \end{aligned}$$

7. Symmetry properties of DFS coefficient:

In the last chapter we discussed some symmetry properties of the discrete time Fourier transform of aperiodic sequence. The same symmetry properties also hold for DFS coefficients and their derivation is also similar in style using linearity, conjugation and time reversal properties DFS coefficients.

8. Time scaling:

Let us define

$$\widetilde{x}_{(m)}[n] = \begin{cases} x[n/m], & \text{if n is multiple of m} \\ 0, \text{if n is not a multiple of m} \end{cases}$$

sequence $\{\widetilde{x}_{(m)}[n]\}$ is obtained by inserting $(m-1)$ zeros between two consecutive values of $\widetilde{x}[n]$. Thus $\{\widetilde{x}_{(m)}[n]\}$ is also periodic, but period is mN. The DFS coefficients are given by

$$\sum_{n=0}^{mN-1} \widetilde{x}_{(m)}[n] e^{-j\frac{2\pi}{mN}kn}$$

putting $n = lm + r,\ 0 \le l \le N-1,\ 0 \le r < m$

$$= \sum_{l=0}^{N-1} \widetilde{x}[l] e^{-j\frac{2\pi}{N}\frac{k}{m}(lm)}$$

as non zero terms occur only when $r = 0$

$$= \widetilde{x}[h].$$

If we define $\widetilde{y}[n] = \widetilde{x}[nM]$ then $\widetilde{y}[n]$ is periodic with period equal to least common multiple (LCM) of M and N. The relationship between DFS coefficients is not simple and we omit it here.

9. Difference

$$\{(\widetilde{x}[n] - \widetilde{x}[n-1])\} \longleftrightarrow \{(1 - e^{-j\frac{2\pi}{N}kn})\widetilde{X}[k]\}$$

This follows from linearity property.

10. Accumulation

Let us define

$$\widetilde{y}[n] = \sum_{k=-\infty}^{n} \widetilde{x}[k]$$

$\{\widetilde{y}[n]\}$ will be bounded and periodic only if the sum of terms of $\widetilde{x}[n]$ over one period is zero, i.e. $\sum_{n=0}^{N-1} \widetilde{x}[n] = 0$, which is equivalent to $\widetilde{X}[0] = 0$. Assuming this to be true

$$\left\{ \sum_{k=-\infty}^{n} \widetilde{x}[k] \right\} \longleftrightarrow \left\{ \left(\frac{1}{1 - e^{-j\frac{2\pi k}{N}}} \right) \widetilde{X}[h] \right\}$$

11. Periodic convolution

Let $\{\widetilde{x}_1[n]\}$ and $\{\widetilde{x}_2[n]\}$ be two periodic signals having same period N with discrete Fourier series coefficients denoted by $\{\widetilde{X}_1[k]\}$ and $\{\widetilde{X}_2[k]\}$ respectively. If we form the product $\widetilde{x}_3[k] = \widetilde{X}_1[k]\widetilde{X}_2[k]$ then we want to find out the sequence

$\{\widetilde{x}_3[n]\}$ whose DFS coefficients are $\{\widetilde{X}_3[k]\}$. From the synthesis equation we have

$$\begin{aligned}\widetilde{x}_3[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}_3[k]e^{j\frac{2\pi}{N}kn}\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}_1[k]\widetilde{X}_2[k]e^{j\frac{2\pi}{N}kn}\end{aligned}$$

substituting for $\widetilde{X}_1[k]$ in terms of $\widetilde{x}_n$ we get

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\widetilde{x}_1[m]e^{-j\frac{2\pi}{N}km}\widetilde{X}_2[k]e^{j\frac{2\pi}{N}kn}$$

interchanging order of summations we get

$$\begin{aligned} &= \sum_{m=0}^{N-1}\widetilde{x}_1[m]\frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}_2[k]e^{j\frac{2\pi}{N}(n-m)k}\\ &= \sum_{m=0}^{N-1}\widetilde{x}_1[m]\widetilde{x}_2[n-m] \qquad (6.15)\end{aligned}$$

as inner sum can be recognized as $\widetilde{x}_2[n-m]$ from the synthesis equation. Thus

$$\sum_{m=0}^{N-1}\widetilde{x}_1[m]\widetilde{x}_2[n-m] \longleftrightarrow \{\widetilde{X}_1[k]\widetilde{X}_2[k]\}$$

The sum in the equation (6.15) looks like convolution sum, except that the summation is over one period. This is known as periodic convolution. The resulting sequence $\{\widetilde{x}_3[n]\}$ is also periodic with period N. This can be seen from equation (6.15) by putting $m+N$ instead of m.
The Duality theorem gives analogous result when we multiply two periodic sequences.

$$\{\widetilde{x}_1[n]\widetilde{x}_2[n]\} \longleftrightarrow \left\{\frac{1}{N}\sum_{l=0}^{N-1}\widetilde{X}_1[l]\widetilde{X}_2[k-l]\right\}$$

The DFS coefficients are obtained by doing periodic convolution of $\{\widetilde{X}_1[k]\}$ and $\{\widetilde{X}_2[k]\}$ and multiplying the result by 1/N. We can also prove this result directly by starting from the analysis equation.
The periodic convolution has properties similar to the aperiodic (linear convolution). It is cumulative, associative and distributes over additions of two signals. The properties of DFS representation of periodic sequence are summarized in the table 6.2

Periodic sequence (period N)	DFS coefficients (Period N)
1. $\{\widetilde{x}[n]\}$	$\{\widetilde{x}[k]\}$ period N
2. $a\{\widetilde{x}[n]\} + b\{\widetilde{y}[n]\}$	$a\{\widetilde{X}[k]\} + b\{\widetilde{Y}[k]\}$
3. $\{\widetilde{x}[n]\}$	$N\{\widetilde{X}[-k]\}$
4. $\{\widetilde{x}[n-m]\}$	$\{e^{-j\frac{2\pi}{N}km}\widetilde{X}[k]\}$
5. $\{e^{j\frac{2\pi}{N}ln}\widetilde{x}[n]\}$	$\{\widetilde{X}[k-l]\}$
6. $\{\widetilde{x}^*[n]\}$	$\{\widetilde{X}^*[-k]\}$
7. $\{x[-n]\}$	$\{\widetilde{X}[-k]\}$
8. $\widetilde{x}_{(m)}[n] = \begin{cases} x[\frac{m}{m}], n = lm \\ 0, \text{otherwise} \end{cases}$ (periodic with period mN)	$\{\widetilde{X}[k]\}$, (Viewed as periodic with period mN)
9. $\{x[n] - x[n-1]\}$	$\{(1 - e^{-j\frac{2\pi}{N}h})\widetilde{X}[k]\}$
10. $\left\{\sum_{m=-\infty}^{n} \widetilde{x}[m]\right\}$ (periodic only if $\widetilde{x}[0] = 0$)	$\left\{\frac{1}{1-e^{-j\frac{2\pi}{N}k}}\widetilde{X}[k]\right\}$
11. $\left\{\sum_{m=0}^{N-1} \widetilde{x}[m]\widetilde{y}[n-m]\right\}$	$\{\widetilde{X}[k]\widetilde{Y}[k]\}$
12. $\{\widetilde{x}[n]\widetilde{y}[n]\}$	$\{\frac{1}{N}\sum_{l=0}^{N-1} \widetilde{X}[l]\widetilde{y}[k-l]\}$
13. $\{Re[\widetilde{x}[n]]\}$	$\{\widetilde{X}_e[k]\} = \{\frac{1}{2}(\widetilde{X}[k] + \widetilde{X}^*[-k])\}$
14. $\{jI_m[\widetilde{x}[n]]\}$	$\{\widetilde{X}_0[k]\} = \{\frac{1}{2}(\widetilde{X}[k] - \widetilde{X}^*[-k])\}$
15. $\{\widetilde{x}_e[n]\} = \{\frac{1}{2}(\widetilde{x}[n] + \widetilde{x}^*[-n])\}$	$\{Re(\widetilde{X}[n])\}$
16. $\{\widetilde{x}_o[n]\} = \{\frac{1}{2}(\widetilde{x}[n] - \widetilde{x}^*[-n])\}$	$\{jI_m[\widetilde{x}[k]]\}$
17. If $\{\widetilde{x}[n]\}$ is real then	$\widetilde{X}[k] = \widetilde{X}^*[-k]$ $Re[\widetilde{X}[k]] = Re[\widetilde{X}[-k]]$ $Im[\widetilde{X}[k]] = -Im[\widetilde{X}[-k]]$ $\lvert\widetilde{X}[k]\rvert = \lvert\widetilde{X}[-k]\rvert$ $\sphericalangle\widetilde{X}[k] = -\sphericalangle\widetilde{X}[-k]$

Fourier Transform of periodic signals: If $\{\widetilde{x}[n]\}$ is periodic with period N, then we can write

$$\{\widetilde{x}[n]\} = \left\{\frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{-j\frac{2\pi}{N}kn}\right\}$$

Using equation (5.9) we see that

$$\begin{aligned}\widetilde{X}(e^{j\omega}) &= \frac{2\pi}{N}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{N-1} \widetilde{X}[k]\delta\left(\omega - \frac{2\pi}{N}k + 2\pi\, l\right) \\ &= \frac{2\pi}{N}\sum_{l=-\infty}^{\infty} \widetilde{X}[l]\delta\left(\omega - \frac{2\pi\, l}{N}\right)\end{aligned}$$

as $\widetilde{X}[h]$ are periodic with period N.

Example: Consider the periodic impulse train

$$\begin{aligned}\widetilde{x}[n] &= \sum_{r=-\infty}^{\infty} \delta[n - rN] \\ \text{then } \widetilde{X}[k] &= \sum_{n=0}^{N-1} \widetilde{x}[n] e^{-j\frac{2\pi}{N}kn} \\ &= 1\end{aligned}$$

as only one term corresponding to $n = 0$ is non zero. Thus the DTFT is

$$\widetilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi\, k}{N}\right)$$

6.2 Fourier Representation of Finite Duration sequence. The Discrete Fourier Transform (DFT):

We now consider the sequence $\{x[n]\}$ such that $x[n] = 0,\ n < 0$ and $x[n] = 0,\ n > N-1$. Thus $x[n]$ can be take non-zero values only for $0 \leq n \leq N-1$. Such sequences are known as finite length sequences, and N is called the length of the sequence. If a sequence has length M, we consider it to be a length N sequence where $M \leq N$. In these cases last $(N - M)$ sample values are zero. To each finite length sequence of length N we can always associate a periodic sequence $\{\widetilde{x}[n]\}$ defined by

$$\widetilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN] \tag{6.16}$$

Note that $\{\widetilde{x}[n]\}$ defined by equation (6.16) will always be a periodic sequence with period N, whether $\{x[n]\}$is of finite length N or not. But when $\{x[n]\}$ has finite length N, we can recover the sequence $\{x[n]\}$ from $\{\widetilde{x}[n]\}$ by defining

$$x[n] = \begin{cases} \widetilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.17}$$

This is because of $\{x[n]\}$ has finite length N, then there is no overlap between terms $x[n]$ and $x[n - mN]$ for different values of $m \neq 0$.
Recall that if
$n = kN + r$, where $0 \leq r \leq N-1$
then $\quad n$ modulo $N = r$,
i.e. we add or subtract multiple of N from n until we get a number lying between 0 to $N-1$. We will use $((n))_N$ to denote n modulo N. Then for finite length sequences of length N equation (6.16) can be written as

$$\widetilde{x}[n] = x[((n))_N] \tag{6.18}$$

We can extract $\{x[n]\}$ from $\{\widetilde{x}[n]\}$ using equation (6.17). Thus there is one-to-one correspondance between finite length sequences $\{x[n]\}$ of length N, and periodic sequences $\{\widetilde{x}[n]\}$ of period N.
Given a finite length sequence $\{x[n]\}$ we can associate a periodic sequence $\{\widetilde{x}[n]\}$

with it.
This periodic sequence has discrete Fourier series coefficients $\{\widetilde{X}[k]\}$, which are also periodic with period N. From equations (6.2) and (6.3) we see that we need values of $\widetilde{x}[n]$ for $0 \leq n \leq N-1$ and $\widetilde{X}[k]$ for $0 \leq k \leq N-1$. Thus we define discrete Fourier transform of finite length sequence $\{x[n]\}$ as

$$X[k] = \begin{cases} \widetilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

where $\{\widetilde{X}[h]\}$ is DFS coefficient of associated periodic sequence $\{\widetilde{x}[n]\}$. From $\{X[k]\}$we can get $\{\widetilde{X}[h]\}$ by the relation.

$$\widetilde{X}[k] = X[((k))_N] = X[(k \, \text{modulo} \, N)]$$

then from this we can get $\{\widetilde{x}[n]\}$ using synthesis equation (6.2) and finally $\{x[n]\}$ using equation (6.17). In equations (6.2) and (6.3) summation interval is 0 to $N-1$, we can write $X[k]$ directly in terms of $x[n]$, and $x[n]$ directly in terms of $X[k]$ as

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} \widetilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}[k] e^{j\frac{2\pi}{N}hn} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}hn}, & 0 \leq N \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

For convenience of notation, we use the complex quantity

$$W_N = e^{-j\frac{2\pi}{N}} \tag{6.19}$$

with this notation, DFT analysis and synthesis equations are written a follows

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \; 0 \leq k \leq N-1 \tag{6.20}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \leq n \leq N-1 \tag{6.21}$$

If we use values of k and n outside the interval 0 to $N-1$ in equation (6.20) and (6.21), then we will not get values zero, but we will get periodic repetition of $X[k]$ and $x[n]$ respectively. In defining DFT, we are concerned with values only in interval 0 to $N-1$. Since a sequence of length M can also be considered a sequence of length N, $N \geq M$, we also specify the length of the sequence by saying N-point-DFT, of sequence $\{x[n]\}$.

Sampling of the Fourier transform:

For sequence $\{x[n]\}$ of length N, we have two kinds of representations, namely, discrete time Fourier transform $X(e^{j\omega})$ and discrete Fourier transform $X[k]$. The DFT values $X[k]$ can be considered as samples of $X(e^{j\omega})$

$$X[k] = \sum_{k=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi}{N}kn}$$

(as $x[n] = 0$ for $n < 0$, and $n > N-1$)

$$= X(e^{j\omega})|_{w=\frac{2\pi}{N}k} \tag{6.22}$$

Thus is $X[k]$ is obtained by sampling $X(e^{jw})$ at $w = \frac{2\pi}{N}k,\ k = 0, 1..N-1$.

Properties of the discrete Fourier transform:

Since discrete Fourier transform is similar to the discrete Fourier series representation, the properties are similar to DFS representation. We use the notation

$$\{x[n]\} \longleftrightarrow \{X[k]\}$$

to say that $\{X[k]\}$ are DFT coefficient of finite length sequence $x[n]$

1. Linearity

If two finite length sequence have length M and N, we can consider both of them with length greater than or equal to maximum of M and N. Thus if

$$\{x[n]\} \longleftrightarrow \{X[k]\}$$

$$\{y[n]\} \longleftrightarrow \{Y[k]\}$$

then

$$a\{x[n]\} + b\{y[n]\} \longleftrightarrow a\{X[k]\} + b\{Y[k]\}$$

where all the DFTs are N-point DFT. This property follows directly from the equation (6.20)

2. Circular shift of a sequence

If we shift a finite length sequence $\{x[n]\}$ of length N, we face some difficulties. When we shift it in right direction $\{x[n-n_0]\}, n_0 > 0$ the length of the sequence will became $(N+n_0)$ according to definition. Similarly if we shift it left $\{x[n-n_0]\}, n_0 < 0$, if may no longer be a finite length sequence as $x[n-n_0]$ may not be zero for $n < 0$. Since DFT coefficients are same as DFS coefficients, we define a shift operation which looks like a shift of periodic sequence. From $\{x[n]\}$ we get the periodic sequence $\{\widetilde{x}[n]\}$ defined by

$$\widetilde{x}[n] = x[((n))_N]$$

We can shift this sequence by m to get

$$\{\widetilde{y}[n]\} = \widetilde{x}[n-m]$$

Now we retain the first N values of this sequence

$$y[n] = \begin{cases} \widetilde{y}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

This operation is shown in figure below for $m = 2$, $N = 5$.
FIGURES

We can see that $\{y[n]\}$ is not a shift of sequence $\{x[n]\}$. Using the properties of the modulo arithmetic we have

$$\widetilde{x}[n-m] = x[((n-m))_N]$$

and

$$y[n] = \begin{cases} x[((n-m))_N]), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.23}$$

The shift defined in equation (6.23) is known as circular shift. This is similar to a shift of sequence in a circular register.
FIGURE

3. Shift property of DFT

From the definition of the circular shift, it is clear that it corresponds to linear shift of the associated periodic sequence and so the shift property of the DFS coefficient will hold for the circular shift. Hence

$$\{x[((n-m))_N],\ 0 \le n \le N-1\} \longleftrightarrow \{W_N^{km} X[k]\} \tag{6.24}$$

and

$$\{W_N^{-nl} x[n]\} \longleftrightarrow \{X[((k-l))_N\ 0 \le k \le N-1\} \tag{6.25}$$

4. Duality

We have the duality for the DFS coefficient given by $\{\widetilde{X}[n]\} \longleftrightarrow \{N\widetilde{X}[-k]\}$, retaining one period of the sequences the duality property for the DFT coefficient will become

$$\{X[n]\} \longleftrightarrow \{N\, x[((-k))_N,\ 0 \le k \le N-1\}$$

5. Symmetry properties

We can infer all the symmetry properties of the DFT from the symmetry properties of the associated periodic sequence $\{\widetilde{x}[n]\}$ and retaining the first period. Thus we have

$$\{\widetilde{x}^*[n]\} \longleftrightarrow \{X^*[((-k))_N],\ 0 \le k \le N-1\}$$

and

$$\{X^*[((-n))_N],\ 0 \le n \le N-1\} \longleftrightarrow \{X^*[k]\}$$

We define conjugate symmetric and anti-symmetric points in the first period 0 to $N-1$ by

$$x_{ep}[n] = \widetilde{x}_e[n] = \frac{1}{2}(x[n] + x^*[((-n))_N]),\ 0 \leq n \leq N-1$$

$$x_{op}[n] = \widetilde{x}_0[n] = \frac{1}{2}(x[n] - x^*[((-n))_N]),\ 0 \leq n \leq N-1$$

Since

$$((-n))_N = \begin{cases} 0 & n = 0 \\ N-n, & 1 \leq n \leq N-1 \end{cases}$$

the above equation similar to

$$x_{ep}[n] = \begin{cases} Re(x[0]), & n = 0 \\ \frac{1}{2}(x[n] - x^*[N-n]), & 1 \leq n \leq N-1 \end{cases} \tag{6.26}$$

$$x_{op}[n] = \begin{cases} jIm(x[0]), & n = 0 \\ \frac{1}{2}(x[n] - x^*[N-n]), & 1 \leq n \leq N-1 \end{cases} \tag{6.27}$$

$$x[n] = x_{ep}[n] + x_{op}[n]$$

$\{x_{ep}[n]\}$ and $\{x_{op}[n]\}$ are referred to as periodic conjugate symmetric and periodic conjugate anti-symmetric parts of $x[n]$. In terms if these sequence the symmetric properties are

$$\begin{aligned} \{Re(x[n])\} &\longleftrightarrow \{X_{ep}[k]\} \\ \{jIm(x[n])\} &\longleftrightarrow \{X_{op}[k]\} \\ \{x_{ep}[n]\} &\longleftrightarrow \{Re(X[k])\} \\ \{x_{op}[n]\} &\longleftrightarrow \{jIm(X[k])\} \end{aligned}$$

6. Circular convolution

We saw that multiplication of DFS coefficients corresponds of periodic convolution of the sequence. Since DFT coefficients are DFS coefficients in the interval, $0 \leq k \leq N-1$, they will correspond to DFT of the sequence retained by periodically convolving associated periodic sequences and retaining their first period.

$$\widetilde{x}[n] = x[((n))_N]$$

$$\widetilde{y}[n] = y[((n))_N]$$

Periodic convolution is given by

$$\widetilde{z}[n] = \sum_{k=0}^{N-1} \widetilde{x}[k]\widetilde{y}[n-k]$$

using properties of the modulo arithmetic

$$\widetilde{z}[n] = \sum_{k=0}^{N-1} x[((k))_N]y[((n-k))_N]$$

and then

$$z[n] = \begin{cases} \widetilde{z}[n], & 0 \leq n \leq N-1 \\ 0, & otherwise \end{cases}$$

Since $((k))_N = k, \qquad 0 \leq k \leq N-1 \qquad$ we get

$$z[n] = \sum_{l=0}^{N-1} x[k]y[((n-k))_N], \qquad 0 \leq n \leq N-1 \tag{6.28}$$

The convolution defined by equation (6.28) is known as N-point-circular convolution of sequence $\{x[n]\}$ and $\{y[n]\}$, where both the sequence are considered sequence of length N. From the periodic convolution property of DFS it is clear that DFT of $\{z[n]\}$ is $\{X[k]Y[k]\}$. If we use the notation
$\{x[n]\} \; \textcircled{N} \; \{y[n]\}$
to denote the N point circular convolution we see that

$$\{x[n]\} \; \textcircled{N} \; \{y[n]\} \longleftrightarrow \{X[h]Y[k]\} \tag{6.29}$$

In view of the duality property of the DFT we have

$$\{x[n]y[n]\} \longleftrightarrow \frac{1}{N}\{X[k]\} \; \textcircled{N} \; \{Y[k]\} \tag{6.30}$$

Properties of the Discrete Fourier transform are summarized in the table 6.2

Table 6.2

Finite length sequence (length N)	N-point DFT (length N)
1. $\{x[n]\}$	$\{X[k]\}$
2. $a\{x[n]\} + b\{y[n]\}$	$a\{X[k]\} + b\{Y[k]\}$
3. $\{X[n]\}$	$N\{X[((-k))_N]\}$
4. $\{x[((n-m))_N]\}$	$\{W_N^{km} X[k]\}$
5. $\{W_N^{-ln} x[n]\}$	$\{X[((k-l))_N]\}$
6. $\{x[n]\} Ⓝ \{y[n]\}$	$\{X[k]Y[k]\}$
7. $\{x[n]y[n]\}$	$\frac{1}{N}\{X[k]\} Ⓝ \{Y[k]\}$
8. $\{x^*[n]\}$	$\{X^*[((-k))_N]\}$
9. $\{x^*[((-n))_N]\}$	$\{X^*[k]\}$
10. $\{Re\, x[n]\}$	$\{X_{ep}[k]\}$
11. $\{jIm(x[n])\}$	$\{X_{op}[k]\}$
12. $\{X_{ep}[n]\}$	$\{Re(X[k])\}$
13. $\{X_{op}[n]\}$	$\{jIm(X[k])\}$
14 If $\{x[n]\}$ is real sequence	$X[k] = X^*[((-k))_N]$
	$Re(X[k]) = Re(X[((-k))_N])$
	$I_m(X[k]) = -I_m(X[((-k))_N])$
	$\lvert X[k]\rvert = \lvert X((-k))_N]\rvert$
	$\measuredangle X[k] = -\measuredangle X[((-k))_N]$

Linear convolution using the Discrete Fourier Transform

Output of a linear time invariant-system is obtained by linear convolution of input signal with the impulse response of the system. If we multiply DFT coefficients, and then take inverse transform we will get circular convolution. From the examples it is clear that result of circular convolution is different from the result of linear convolution of two sequences. But if we modify the two sequence appropriately we can get the result of circular convolution to be same as linear convolution. Our interest in doing linear convolution results form the fact that fast algorithms for computing DFT and IDFT are available. These algorithms will be discussed in a later chapter. Here we show how we can make result of circular convolution same as that of linear convolution.
If we have sequence $\{x[n]\}$ of length L and a sequence $\{y[n]\}$ of length M, the sequence $\{z[n]\}$ obtained by linear convolution has length $(L + M - 1)$. This can be seen from the definition.

$$
\begin{aligned}
Z[n] &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] \\
&= \sum_{k=0}^{L-1} x[k]y[n-k] \qquad (6.31)
\end{aligned}
$$

as $x[k] = 0$ for $k < 0$, $k \geq L$. For $n < 0$, $y[n-k] = 0$, $0 \leq k \leq L-1$ hence $z[n] = 0$. Similarly for $n > L + M - 2$, $y[n-k] = 0$, $0 \leq k \leq L-1$, so $z[n] = 0$, $n > L + M - 2$. Hence $z[n]$ is possibly nonzero only for $0 \leq n \leq L + M - 2$. Now consider a sequence $\{w[n]\}$, DTFT is given by

$$
W(e^{jw}) = \sum_{n=-\infty}^{\infty} w[n]e^{-jwn}
$$

writing $n = mN + l$, $-\infty < m < \infty$, $0 \leq l \leq N-1$
We get

$$W(e^{jw}) = \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} w[mN+l]e^{-jw(mN+l)}$$

If we take $w = \frac{2\pi}{N}k$, we see that

$$W(e^{j\frac{2\pi}{N}k}) = \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} W[l+mN]e^{-j\frac{2\pi}{N}l}$$

Comparing this with the DFT equation (6.), we see that $W(e^{j\frac{2\pi}{N}k})$ can be seen as DFT coefficients of a sequence

$$v[l] = \sum_{m=-\infty}^{\infty} w[l+mN], \qquad 0 \leq l \leq N-1 \tag{6.32}$$

obviously if $\{w[n]\}$ has length less then or equal to N, then

$$v[l] = w[l], \qquad 0 \leq l \leq N-1$$

However, if the length of $\{W[n]\}$ is greater than N, $v[l]$ may not be equal to $w[l]$ for all values of l.
The sequence $\{z[n]\}$ in equation (6.31) has the discrete Fourier transform

$$Z(e^{jw}) = X(e^{jw})Y(e^{jw})$$

The N-point DFT of $\{z[n]\}$ sequence is

$$\begin{aligned} Z[k] &= Z(e^{j\frac{2\pi}{N}k}) \\ &= X(e^{j}\frac{2\pi}{N}k)Y(e^{j}\frac{2\pi}{N}k) \\ &= X[k]Y[k] \end{aligned}$$

where $\{X[k]\}$ and $\{Y[k]\}$ are N-point DFTs of $\{x[n]\}$ and $\{y[n]\}$ respectively.
The sequence resulting as the inverse DFT of $Z[k]$ is then by equation (6.32).

$$v[n] = \begin{cases} \sum_{m=-\infty}^{\infty} z[n+mN], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

From the circular convolution property of the DFT we have

$$\{v[n]\} = \{x[n]\} \circledN \{y[n]\}$$

Thus, the circular convolution of two-finite length sequences can be viewed as linear convolution, followed time aliasing, defined by equation (6.32). If N is greater than or equal to $(L+M-l)$, then there will be no time aliasing as the linear convolution produces a sequence of length $(L+M-l)$. Thus we can use circular convolution for linear convolution by padding sufficient number of zeros at the end of a finite length sequence. We can use DFT algorithm for calculating the circular convolution.